Soln. of C5

1. $f(z) = z^n$, $n = 0, 1, 2, \cdots$

$f(z) = \sum_{n=0}^{\infty} a_n z^n \Rightarrow a_0 = f(0) = f(0)^2 = a_0^2 \Rightarrow a_0 = 0$ or $a_0 = 1$.

Suppose $a_0 = 1$ and f is nonconstant. Let k be the smallest positive integer s.t $a_k \neq 0$, then $f(z^2) = 1 + a_k z^{2k} +$ higher order terms and $f^2(z) = 1 + 2a_k z^k +$ higher order terms $\Rightarrow$ as $f(z^2) = f(z)^2$ that $2a_k = 0 \Rightarrow a_k = 0$ — contradiction. So f is constant.

Suppose that $a_0 = 0$ and f is nonconstant. Let k be as above, then $f(z) = z^k(a_k + a_{k+1} z + a_{k+2} z^2 + \cdots) = z^k g(z)$ and g is entire. Now, $f(z^2) = f(z)^2 \Rightarrow z^{2k} g(z^2) = z^{2k} g(z)^2$ $\Rightarrow g(z^2) = g(z)^2 \Rightarrow a_k = 1$ (as k is smallest s.t. $a_k \neq 0$) $\Rightarrow g(z) = 1$ $\forall z$ by the previous case $\Rightarrow f(z) = z^k$.

2. a) Use differentiation.

b) If $g(z) = 1 - \frac{1}{z}$ then $g'(z) = f(z)$ and use the geometric series for $\frac{1}{z} = \frac{1}{(z-1)\left(1 + \frac{1}{z-1}\right)}$ using $\frac{1}{|z-1|} < 1$.

3. a) $\frac{5}{4} < |z| < \frac{3}{2}$, $f(z) = \frac{6z+8}{(2z+3)(4z+5)} = \frac{1}{2z+3} + \frac{1}{4z+5}$

$$= \frac{1}{3\left(1+\frac{2z}{3}\right)} + \frac{1}{4z\left(1+\frac{5}{4z}\right)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{3^{n+1}} z^n + \sum_{n=0}^{\infty} \frac{(-1)^n 5^n}{4^{n+1}} \cdot \frac{1}{z^{n+1}}$$

$|z| < \frac{5}{4}$, $f(z) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{2^n}{3^{n+1}} + \frac{5^{-n}}{4^{n+1}} \right) z^n$

$|z| > \frac{3}{2}$, $f(z) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{3^n}{2^{n+1}} + \frac{5^{-n}}{4^{n+1}} \right) z^{-(n+1)}$

b) $f(z) = \frac{1}{z^3(1-z)}$

$|z| < 1$, $f(z) = \frac{1}{z^3}(1 + z + z^2 + \cdots) = \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + z^3 + \cdots$

$|z| > 1$, $f(z) = -\frac{1}{z^4\left(1 - \frac{1}{z}\right)} = -\frac{1}{z^4}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right)$

4. $e^{z+\frac{1}{z}} = \left(\sum_{k=0}^{\infty}\frac{z^k}{k!}\right)\left(\sum_{j=0}^{\infty}\frac{1}{j!\,z^j}\right) = \sum_{n=-\infty}^{\infty}\left(\sum_{\substack{k-j=n\\ k,j\geq 0}}\frac{1}{k!\,j!}\right)z^n$

$$= \sum_{n=-\infty}^{\infty}\left(\sum_{j\geq \max(0,-n)}\frac{1}{j!\,(n+j)!}\right)z^n$$

$\Rightarrow a_n = \sum_{j\geq \max(0,-n)}\frac{1}{j!\,(n+j)!}$

By uniqueness of Laurent-series, $a_n = \int_C \frac{f(w)}{w^{n+1}}\,dw \cdot 2\pi i$ where C is the unit circle.

$\Rightarrow a_n = \frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}\cos n\theta\, d\theta$ (imaginary part is zero as a_n is real).

$\Rightarrow$ For $n\geq 0$, $\frac{1}{2\pi}\int_0^{2\pi} e^{2\cos\theta}\cos n\theta\, d\theta = \sum_{j=0}^{\infty}\frac{1}{j!\,(n+j)!}$

5. No. Let $P(z) = a_0 + a_1 z + \cdots + a_m z^n$, $e^{1/z} = \sum_{n=0}^{\infty}\frac{1}{n!}\cdot\frac{1}{z^n}$. Then the Laurent-series of $P(z)e^{1/z}$ has terms of the form $\frac{C_N}{z^N}$ (around zero) where $C_N \neq 0$, $N > 0$.

coefficient of $\frac{1}{z^N}$: $\frac{a_0}{N!} + \frac{a_1}{(N+1)!} + \cdots + \frac{a_m}{(N+m)!}$

Let r be the smallest nonnegative integer s.t $a_r \neq 0$ then $C_N = \frac{1}{(N+r)!}\left[a_r + \frac{a_{r+1}}{N+r+1} + \cdots + \frac{a_m}{(N+r+1)\cdots(N+m)}\right] \neq 0$

for large Ns as $f(N) = \frac{a_{r+1}}{N+r+1} + \cdots + \frac{a_m}{(N+r+1)\cdots(N+m)}$

is nonconstant function.

6. a) $\frac{\sin z}{z^2-\pi^2} = \frac{-\sin(z-\pi)}{(z-\pi)(z+\pi)}$ → removable singularity

b) $\frac{\sin z}{(z-\pi)^2} = -\frac{\sin(z-\pi)}{(z-\pi)^2}$ → simple pole as $\lim_{z\to\pi}\frac{\sin(z-\pi)}{z-\pi} \neq 0$.

c) $\frac{z\cos z}{1-\sin z} = \frac{z(1-\sin^2 z)}{\cos z(1-\sin z)} = \frac{z(1+\sin z)}{\cos z} = \frac{z(1+\sin z)}{(z-\pi/2)\,h(z)}$, $h(\pi/2)\neq 0$.

$$= \frac{1+\sin z}{h(z)} + \frac{\pi/2\,(1+\sin z)}{(z-\pi/2)\,h(z)} \rightarrow \text{simple pole.}$$