Lectures 13: Comparison, Limit Comparison and Cauchy Condensation Tests

In this and the next lecture, we consider certain tests which deal with the absolute convergence or divergence of series. In this lecture, we discuss an important test called comparison test. The other tests which will be discussed will follow from the comparison test.

We will use the following elementary result.

Theorem 13.1. *Let $a_n \geq 0$ for all n. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if (S_n) is bounded.*

Proof. Since $a_n \geq 0$ for all n, (S_n) is an increasing sequence and $S_n \geq 0$ for all n. Hence (S_n) converges if and only if (S_n) is bounded, which proves the result. $\square$

Comparison test

Comparison test determines the convergence or the divergence of a series by comparing it to the one whose behavior is already known.

Theorem 13.2 (Comparison test). *Let $0 \leq a_n \leq b_n$ for all n.*

(1) *If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.*

(2) *If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.*

Proof. (1) Suppose $\sum_{n=1}^{\infty} b_n$ converges. Then the sequence of partial sums of $\sum_{n=1}^{\infty} b_n$ is bounded. Since $0 \leq a_n \leq b_n$ for all n, the sequence of partial sums $\sum_{n=1}^{\infty} a_n$ is bounded. Now, by Theorem 13.1, $\sum_{n=1}^{\infty} a_n$ converges.

(2) If $\sum_{n=1}^{\infty} b_n$ converges, then by (1), $\sum_{n=1}^{\infty} a_n$ converges which is a contradiction. $\square$

Example 13.3. 1. The series $\sum_{n=1}^{\infty} \frac{1}{(n+1)^2}$ converges because $\frac{1}{(n+1)(n+1)} \leqslant \frac{1}{n(n+1)}$. By Theorem 12.9, $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

2. The series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges because $\frac{1}{n} \leqslant \frac{1}{\sqrt{n}}$.

3. The series $\sum_{n=1}^{\infty} \frac{\sqrt{n^4-3}}{n^4+7}$ converges, because, $\frac{\sqrt{n^4-3}}{n^4+7} \leq \frac{\sqrt{n^4}}{n^4}$.

4. The series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges because $\frac{1}{n!} < \frac{1}{2^{n-1}}$.

5. Consider $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{(\sin n)(\cos n)}{e^n}$. To apply the comparison test, the terms have to be non-negative. Note that $|a_n| \leq \frac{1}{e^n}$. Since $\sum_{n=1}^{\infty} \frac{1}{e^n}$ converges, by the comparison test, the series $\sum_{n=1}^{\infty} |a_n|$ converges. By Theorem 12.6, $\sum_{n=1}^{\infty} a_n$ converges.

Limit Comparison Test

To apply the comparison test, the given series needs to be compared to another series whose convergence/divergence is already known. In general, finding a suitable series for comparison can be difficult or tricky. Let us illustrate with a simple example. Consider the series $\sum_{n=4}^{\infty} \frac{1}{n^2-5n+6}$. This series cannot be directly compared with the series such as $\frac{1}{n^2}$ or $\frac{1}{n}$. However, observe that

$$\frac{1}{n^2-5n+6} = \frac{1}{(n-3)(n-2)} \leq \frac{1}{(n-3)^2}$$

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which allows us to use the comparison test. The limit comparison test, which is user friendly, is a modification of the comparison test.

Theorem 13.4 (Limit Comparison Test). *Let $a_n \geq 0$ and $b_n > 0$ for all n. Suppose $\frac{a_n}{b_n} \to L$.*

(1) *If $L \in \mathbb{R}$ and $L > 0$, then both $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} a_n$ converge or diverge together.*

(2) *If $L = 0$, and $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges.*

(3) *If $L = \infty$ and $\sum_{n=1}^{\infty} b_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.*

Proof. (1) Suppose $L > 0$. Choose some $\epsilon > 0$ such that $L - \epsilon > 0$. Since $\frac{a_n}{b_n} \to L$ there exists n_0 such that $L - \epsilon < \frac{a_n}{b_n} < L + \epsilon$ for all $n \geq n_0$. Hence $(L-\epsilon)b_n < a_n < (L+\epsilon)b_n$ for all $n \geq n_0$. Now, (1) follows from the comparison test.

(2) Since $\frac{a_n}{b_n} \to 0$, there exists n_0 such that $\frac{a_n}{b_n} < 1$ for all $n > n_0$. This implies that $a_n < b_n$ for all $n > n_0$. Apply the comparison test.

3. Since $\frac{a_n}{b_n} \to \infty$, there exists n_0 such that $\frac{a_n}{b_n} > 1$ for all $n > n_0$. Hence $a_n > b_n$ for all $n > n_0$. Use the comparison test. $\square$

Example 13.5. 1. Consider the series $\sum_{n=4}^{\infty} a_n$ where $a_n = \frac{1}{n^2-5n+6}$. This series is already considered above. We will show the convergence using the limit comparison test (in short, LCT). We need to find (b_n) such that $\frac{a_n}{b_n} \to L$ for some $L \geq 0$ and $\sum_{n=1}^{\infty} b_n$ converges. In this problem, it is easy to guess (b_n), because, as n grows to infinity, the most dominating term in the denominator is n^2. This intuition suggest to choose $b_n = \frac{1}{n^2}$ for all n. Indeed, $\frac{a_n}{b_n} \to 1$. Hence by the LCT, $\sum_{n=4}^{\infty} a_n$ converges.

2. Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{n^2+\sin^2 n}{1+\sqrt{n^5}+\cos n}$. Observe that $a_n \geq 0$ for all n. As guessed in the preceding example, the intuition suggest to choose $b_n = \frac{n^2}{\sqrt{n^5}}$. Verify that $\frac{a_n}{b_n} \to 1$. Therefore, by the LCT, $\sum_{n=1}^{\infty} a_n$ diverges.

3. Let $a_n = \frac{(n+3)3^n}{(2n+1)5^n}$ and consider $\sum_{n=1}^{\infty} a_n$. In this case, a guess is that as n is large, a_n behaves a bit like $b_n = \frac{3^n}{5^n}$. In fact, $\frac{a_n}{b_n} \to \frac{1}{2}$. Hence by the LCT, $\sum_{n=1}^{\infty} a_n$ converges.

4. Let $a_n = 1 - n\sin\frac{1}{n}$ and consider $\sum_{n=1}^{\infty} a_n$. First, observe that $a_n \geq 0$ for all n as $\sin\frac{1}{n} \leq \frac{1}{n}$. By Taylor's theorem, there exists $c \in (0, \frac{1}{n})$ such that

$$\sin\frac{1}{n} = \frac{1}{n} - \frac{1}{3!}\left(\frac{1}{n}\right)^3 + \frac{\cos c}{5!}\left(\frac{1}{n}\right)^5.$$

This implies that $1 - n\sin\frac{1}{n} = \frac{1}{6}\frac{1}{n^2} - \frac{\cos c}{5!}\frac{1}{n^4}$ which implies that $\frac{1-nsin\frac{1}{n}}{\frac{1}{n^2}} \to \frac{1}{6}$ (How ? See below). Hence, in this case we can take $b_n = \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty} b_n$ converges, by LCT, $\sum_{n=1}^{\infty} a_n$ converges.

The above method leads us to guess b_n as follows. If we use the symbol $\approx$ for approximately equal to, then by Taylor's theorem, we can consider either $\sin\frac{1}{n} \approx \frac{1}{n}$ or $\sin\frac{1}{n} \approx \frac{1}{n} - \frac{1}{3!}\left(\frac{1}{n}\right)^3$. In this case, we opt for the second one as the first one does not serve the purpose. Hence, we consider, $1 - n\sin\frac{1}{n} \approx \frac{1}{6n^2}$. This process leads us to guess $b_n = \frac{1}{6n^2}$. For finding $\lim_{n\to\infty} \frac{1-nsin\frac{1}{n}}{\frac{1}{6n^2}}$, we may use the L'hospital rule as follows. Note that

$$\lim_{x\to\infty} \frac{1 - x\sin\frac{1}{x}}{\frac{1}{6x^2}} = \lim_{x\to 0^+} \frac{1 - \frac{1}{x}\sin x}{\frac{x^2}{6}} = \lim_{x\to 0^+} \frac{6(x - \sin x)}{x^3} = 1.$$

We used the L'hospital rule to obtain the last equality in the preceding equation. Since

$$\lim_{x\to\infty} \frac{1 - x\sin\frac{1}{x}}{\frac{1}{6x^2}} = 1,$$

by the definition of the limit of a function (see Lecture 6), $\lim_{n\to\infty} \frac{1-nsin\frac{1}{n}}{\frac{1}{6n^2}} = 1$.

5. Consider the series $\sum_{n=1}^{\infty} \sin\frac{1}{n}$. In this case, we may try with $\sin\frac{1}{n} \approx \frac{1}{n}$. Now, $\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1$. By the LCT, the given series diverges.

6. Consider $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{1}{n}\ln(1+\frac{1}{n})$. In this case, we may try with $\ln(1+x) \approx x$. So, $\frac{1}{n}\ln(1+\frac{1}{n}) \approx \frac{1}{n^2}$. Choose $b_n = \frac{1}{n^2}$. Now, $\lim_{n\to\infty} \frac{\frac{1}{n}\ln(1+\frac{1}{n})}{\frac{1}{n^2}} = \lim_{x\to 0^+} \frac{\ln(1+x)}{x} = 1$. Hence, by the LCT, the given series converges.

Cauchy condensation test

So far, we have not discussed the behavior of the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $1 < p < 2$. We will obtain the convergence of this series as a consequence of the Cauchy condensation test.

Theorem 13.6 (**Cauchy condensation test**). *If $a_n \geq 0$ and $a_{n+1} \leq a_n$ for all n, then $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges.*

Proof (*). Let $S_n = a_1 + a_2 + \cdots + a_n$ and $T_k = a_1 + 2a_2 + \cdots + 2^k a_{2^k}$ for every n and k in $\mathbb{N}$.

Suppose $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges, i.e., (T_k) converges as $k \to \infty$. Hence, there exists M such that $T_k \leq M$ for all $k \in \mathbb{N}$. For a fixed n, choose k such that $2^k \geq n$. Then,

$$\begin{aligned}
S_n &= a_1 + a_2 + \cdots + a_n \\
&\leq a_1 + (a_2 + a_3) + \cdots + (a_{2^k} + \cdots + a_{2^{k+1}-1}) \\
&\leq a_1 + 2a_2 + \cdots + 2^k a_{2^k} \\
&= T_k \\
&\leq M.
\end{aligned}$$

This shows that (S_n) is bounded above and hence $\sum_{n=1}^{\infty} a_n$ converges.

Suppose $\sum_{n=1}^{\infty} a_n$ converges, i.e., (S_n) converges. For a fixed k, choose n such that $n \geq 2^k$. Then

$$\begin{aligned}
S_n &= a_1 + a_2 + \cdots + a_n \\
&\geq a_1 + a_2 + (a_3 + a_4) + \cdots + (a_{2^{k-1}+1} + \cdots + a_{2^k}) \\
&\geq \frac{1}{2}a_1 + a_2 + 2a_4 + \cdots + 2^{k-1} a_{2^k} \\
&= \frac{1}{2}T_k.
\end{aligned}$$

This shows that (T_k) is bounded above and hence (T_k) converges, i.e., $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges. □

Example 13.7. 1. We show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$. We have seen in Example 12.4 that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ diverges if $p \leq 0$. Suppose $p > 0$. To apply Theorem 13.6, consider

$$\sum_{k=0}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=0}^{\infty} 2^k \frac{1}{2^{kp}} = \sum_{k=0}^{\infty} 2^{(1-p)k} = \sum_{k=0}^{\infty} \left[2^{(1-p)}\right]^k.$$

Since the series given above is geometric, it converges if and only if $2^{(1-p)} < 1$, i.e., $p > 1$.

2. We show that $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ converges if and only if $p > 1$. To apply Theorem 13.6, consider $\sum_{k=1}^{\infty} 2^k \frac{1}{2^k(\ln 2^k)^p} = \sum_{k=1}^{\infty} \frac{1}{k^p(\ln 2)^p}$. Now, by the preceding example $\sum_{n=2}^{\infty} \frac{1}{k^p(\ln 2)^p}$ converges if and only if $p > 1$, as $(\ln 2)^p$ is a constant for a given p.

3. Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{\ln n}{n^p}$ and $p \in \mathbb{R}$. Suppose $p \leq 1$. Note that $a_n \geq \frac{1}{n^p}$ for $n \geq 2$. Hence by the comparison test $\sum_{n=1}^{\infty} a_n$ diverges for $p \leq 1$. Suppose $p > 1$. In this case, if we want to use the LCT, then $\ln x \approx x$ may not help for guessing b_n. Note that for any fixed $p > 1$, the series $\sum_{n=1}^{\infty} a_n$ has to either converge or diverge. So, a candidate for b_n is $\frac{1}{n^q}$ for some q. This suggests us to choose $b_n = \frac{1}{n^q}$, where q is unknown, and apply the LCT. Now,

$$\frac{a_n}{b_n} = \frac{\ln n/n^p}{1/n^q} = \frac{\ln n}{n^{p-q}}.$$

If we choose q such that $1 < q < p$, in particular $q = \frac{1+p}{2}$, then, by the L'Hospital rule, $\frac{a_n}{b_n} \to 0$. Since $\sum_{n=1}^{\infty} \frac{1}{n^q}$ converges, the series $\sum_{n=1}^{\infty} a_n$ converges for $p > 1$.