Lecture 15: Power Series, Taylor Series

In one of the previous lectures (Lecture 12), we asked a question whether, for a given $x \in \mathbb{R}$, the series $\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ converges to e^x. This question will be answered at the end of this lecture. Observe that the $(n+1)$-th term of $\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ is $\frac{1}{n!}x^n$ where $\frac{1}{n!}$ is fixed and x varies in $\mathbb{R}$. Such a type of series is called a power series. Let us formally define a power series.

Power series

Let $a_n \in \mathbb{R}$ for $n = 0, 1, 2, \ldots$. The series $\sum_{n=0}^{\infty} a_n x^n$, $x \in \mathbb{R}$, is called a power series. More generally, if $c \in \mathbb{R}$, then the series $\sum_{n=0}^{\infty} a_n (x-c)^n$, $x \in \mathbb{R}$, is called a power series at c. If we take $y = x - c$ then the power series at c reduces to $\sum_{n=0}^{\infty} a_n y^n$ which is a power series at 0. In this lecture we discuss the convergence of power series.

Examples 15.1. 1. Consider the power series $\sum_{n=0}^{\infty} a_n x^n$ where $a_n = \frac{1}{n!}$ for all n. Let us apply the ratio test and find the set of points in $\mathbb{R}$ on which the series converges. For any $x \in \mathbb{R} \setminus \{0\}$, $\frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \frac{|x|}{n+1} \to 0$ as $n \to \infty$. Therefore, the series $\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ converges absolutely for all $x \in \mathbb{R}$.

2. We know that the geometric series $\sum_{n=0}^{\infty} x^n$ converges only in $(-1, 1)$.

3. Using the ratio test, it is easy to verify that the series $\sum_{n=0}^{\infty} n!x^n$ converges only at $x = 0$.

The following result gives an idea about the set on which a power series converges.

Theorem 15.1. *Suppose $\sum_{n=0}^{\infty} a_n x^n$ converges for some $x_0 \neq 0$ and diverges for some x_1. Then*

(i) $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for all x such that $|x| < |x_0|$;

(ii) $\sum_{n=0}^{\infty} a_n x^n$ diverges for all x such that $|x| > |x_1|$.

Proof (*). (i) Suppose $\sum_{n=0}^{\infty} a_n x_0^n$ converges for some $x_0 \neq 0$ and $|x| <| x_0 |$. Since $a_n x_0^n \to 0$, there exists $M \in \mathbb{R}$ such that $|a_n x_0^n| \leq M$ for all $n \in \mathbb{N}$. Therefore,

$$|a_n x^n| = |a_n x_0^n| \left|\frac{x}{x_0}\right|^n \leq M \left|\frac{x}{x_0}\right|^n \quad \text{for all} \quad n \in \mathbb{N}.$$

Since $|\frac{x}{x_0}| < 1$, by the comparison test, the series $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.

(ii) Let $x \in \mathbb{R}$ and $|x| > |x_1|$. Suppose $\sum_{n=0}^{\infty} a_n x^n$ converges. Then by (i), the series $\sum_{n=0}^{\infty} a_n x_1^n$ converges absolutely which is a contradiction. □

For a given power series $\sum_{n=0}^{\infty} a_n x^n$, define $S = \{x \in \mathbb{R} : \sum_{n=0}^{\infty} a_n x^n \text{ is convergent}\}$. If S is bounded then observe from Theorem 15.1 that $\sup S$ can be either 0 or r for some $r > 0$. Hence, it follows again from Theorem 15.1 (see Problem 1 of PP15) that the possibilities for S are

$$\{0\}, \quad \mathbb{R}, \quad (-r, r), \quad [-r, r), \quad (-r, r] \text{ and } [-r, r] \quad \text{for some } r > 0.$$

If S is $\mathbb{R}$ (resp., $\{0\}$) then we say that the radius of convergence of the power series is ∞ (resp., 0). In case S is an interval of the form $(-r, r), [-r, r), (-r, r], [-r, r]$ for some $r > 0$, then the radius of convergence is r.

Examples 15.2. 1. We have already seen above that the power series $\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ converges for all $x \in \mathbb{R}$ and hence the radius of convergence is ∞. Similarly, the radius of convergence of $\sum_{n=0}^{\infty} n!x^n$ (resp., $\sum_{n=0}^{\infty} x^n$) is 0 (resp., 1).

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2. Consider the power series $\sum_{n=0}^{\infty} \frac{x^n}{n}$. Let us apply the ratio test to find the radius of convergence. Since $a_n = \frac{1}{n}$ for all n, for $x \in \mathbb{R} \setminus \{0\}$, we have

$$\frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \left|\frac{x^{n+1}n}{(n+1)x^n}\right| = \left|\frac{n}{n+1}x\right| \to |x| \text{ as } n \to \infty.$$

It is clear that the series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. Therefore, the radius of convergence is 1. Since the series converges for $x = -1$ by the Leibniz test and diverges for $x = 1$, the set S in this case is $[-1, 1)$. Similarly, the set S corresponding to the power series $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ is $[-1, 1]$.

3. Consider the power series $\sum_{n=0}^{\infty} \frac{(x-2)^n}{n3^n}$. We will use the root test to determine the set in which the given series converges. For a given x, let $d_n = \frac{(x-2)^n}{n3^n}$. Then $|d_n|^{1/n} \to |\frac{x-2}{3}|$. By the root test, the given series converges for all x satisfying $|x-2| < 3$, i.e., $-1 < x < 5$ and diverges for all x satisfying $|x - 2| > 3$. We need to check the convergence/divergence for $x = -1$ and $x = 5$. At $x = -1$, the series converges by the Leibniz test and the series diverges when $x = 5$. Hence the series converges only on $[-1, 5)$. Since the given power series is at 2, the radius of convergence is considered to be 3.

Let us come back to the question which was posed in the beginning of this lecture.

Taylor's Series

Taylor's polynomial which was introduced when we discussed Taylor's theorem, leads to the Taylor series. Let $D \subset \mathbb{R}$, $c \in D$, $f : D \to \mathbb{R}$ and $f^{(n)}(c)$ exist for all n. The nth degree polynomial $P_n(x)$ defined by

$$P_n(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is called the Taylor polynomial (with respect to f and c). The power series

$$f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \cdots \qquad \left(\text{denoted by} \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n\right)$$

is called the Taylor series of f at c. If $c = 0$, then the Taylor series of f at c is called Maclaurin's series.

Remark 15.1. 1. Observe from the definitions of the Taylor polynomial $P_n(x)$ and the Taylor series (defined above) that $P_n(x)$ is a partial sum of the Taylor series.

2. In order to compute the Taylor series of f at c, we need only $f^{(n)}(c)$ for every n.

Example 15.3. 1. Let $f(x) = e^x$ for all $x \in \mathbb{R}$. Since $f^{(n)}(x) = e^x$ for all n, the Taylor series of f at 0 is $\sum_{n=0}^{\infty} \frac{1}{n!}x^n$ whereas the Taylor series of f at 1 is $\sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n$.

2. Let $f(x) = \frac{1}{x}, x \neq 0$. Observe that for $x \neq 0$, $f^{(n)}(x) = \frac{(-1)^n n!}{x^{n+1}}$ for all n. It follows that the Taylor series of f at 2 is $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}}(x-2)^n$ and at 3 is $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}}(x-3)^n$

Convergence of Taylor's series

Taylor's theorem is needed to discuss the convergence of Taylor's series. In the rest of this lecture, we assume $f : (a, b) \to \mathbb{R}$ to be infinite times differentiable, where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in$

$\mathbb{R} \cup \{\infty\}$ and $a < b$. For every $x, c \in (a, b)$ and $n \in \mathbb{N}$, by Taylor's theorem, there exists c_n between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-c)^{n+1}.$$

One may get an impression from the preceding equation that the Taylor series of f at c, which is a power series, may converge to $f(x)$ for every $x \in (a, b)$. We will see below that the Taylor series of f may not converge for all $x \in (a, b)$ and even if it converges for some x, it need not converge to $f(x)$.

Example 15.4. 1. If we consider the function $g_1 : \mathbb{R}\backslash\{1\} \to \mathbb{R}$ given by $g_1(x) = 1/(1-x)$, then the Maclaurin series of f is the geometric series $\sum_{n=0}^{\infty} x^n$ which converges only on $(-1, 1)$. Observe that the function g_1 is infinite times differentiable on $(-\infty, 1)$ but the Maclaurin series of f does not converge to $g_1(x)$ for any $x \in (-\infty, -1]$.

2. Define $g_2(x) = e^{-1/x^2}$ for $x \neq 0$ and $g_2(0) = 0$. Using the L'Hospital rule, we can show (see Problem 5 of PP15) that $g_2^{(n)}(0) = 0$ for all $n = 1, 2, ...$ Therefore the Maclaurin series of g_2 (for any $x \in \mathbb{R}$) is identically zero and it does not converge to $g_2(x)$ at any $x \neq 0$.

We now answer the important question that under what condition on f, the Taylor series of f converges to $f(x)$ when x is in the domain of the convergence of the given Taylor's series. Let $x, c \in (a, b)$. By Taylor's theorem, there exists c_n between x and c such that

$$f(x) - P_n(x) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-c)^{n+1}$$

Note that c_n depends also on n. Denote $E_n(x) = f(x) - P_n(x)$. It is clear that the Taylor series of f at c converges to $f(x)$ if and only if $E_n(x) \to 0$ (as $(P_n(x))$ is the sequence of partial sums of the Taylor series). We use this characterization below to show the convergence of Taylor's series of some common functions.

Example 15.5. 1. Let $f(x) = \sin x, x \in \mathbb{R}$. We will show that the Maclaurin series of f converges to $f(x)$ at every $x \in \mathbb{R}$. Fix $x \in \mathbb{R}$. By Taylor's theorem, there exists c_n between 0 and x such that

$$E_n(x) = \frac{f^{(n+1)}(c_n)}{(n+1)!}(x-0)^{n+1}.$$

Since $\mid f^{(n)}(t) \mid \leq 1$ for all $n \in \mathbb{N}$ and all $t \in \mathbb{R}$, observe that $|E_n(x)| \leq \left|\frac{x^{n+1}}{(n+1)!}\right|$. Let $d_n = \frac{x^n}{n!}$ for every n. Then $|\frac{d_{n+1}}{d_n}| \to 0$. By the ratio test for sequence, $|\frac{x^{n+1}}{(n+1)!}| \to 0$ and hence $|E_n(x)| \to 0$. Therefore, the Macluarin series of f converges to $f(x)$ at all $x \in \mathbb{R}$. So, we can expand the sin function in a series form on the whole of $\mathbb{R}$ and we write $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ for every $x \in \mathbb{R}$.

Similarly, we can show that $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ for every $x \in \mathbb{R}$.

2. We will show that $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for all $x \in \mathbb{R}$. Let $f(x) = e^x$ for all $x \in \mathbb{R}$. Fix $x \in \mathbb{R}$. If $x = 0$, then there is nothing to show. Suppose $x \neq 0$. By Taylor's Theorem, there exists c_n between 0 and x such that

$$|E_n(x)| = \left|\frac{f^{n+1}(c_n)}{(n+1)!} x^{n+1}\right| = \left|\frac{e^{c_n}}{(n+1)!} x^{n+1}\right|.$$

Since c_n depends on n, we bypass c_n as follows. If $x > 0$, then $e^{c_n} \leq e^x$ and if $x < 0$ then $e^{c_n} \leq e^0$. Therefore, $e^{c_n} \leq e^{|x|}$ and hence $|E_n(x)| \leq |\frac{e^{|x|}}{(n+1)!} |x^{n+1}|$ for all n. Let $d_n = \frac{e^{|x|}}{n!} x^n$ for all n. Then

$|\frac{d_{n+1}}{d_n}| = |\frac{x}{n+1}| \to 0$. This implies that $d_n \to 0$ and hence $E_n(x) \to 0$. Therefore, $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$ for all $x \in \mathbb{R}$.

Convergence of Maclaurin series of $\ln(1 + x)$ is discussed in Problems 3 and 6 of PP15.