Lecture 16: Riemann Integration (Part I)

At the high school level the indefinite and definite integrals are introduced as follows. For a given function f if there exists F such that $F'(x) = f(x)$ for all x in the domain of f, then the indefinite integral $\int f(x)dx$ is defined to be $F(x) + C$ where C is a constant. Whereas, if f is continuous on $[a, b]$, then the definite integral $\int_a^b f(x)dx$ is defined (but not in a rigorous manner) as the area of the region bounded by the curve $y = f(x)$, $a \leq x \leq b$, the x-axis and the ordinates $x = a$ and $x = b$. Usually, at the school level the following important result called Fundamental Theorem of Calculus (FTC), which enables us to evaluate definite integrals by making use of the indefinite integral, is stated without proof.

Theorem (FTC). *If $f : [a, b] \to \mathbb{R}$ is continuous and $F'(x) = f(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$.*

In this course, we define the definite integral (for functions which need not be continuous) in a rigorous manner and prove a stronger form of the FTC. We will not discuss the methods of evaluating the indefinite integrals as they are covered in the school level. However, we will present some applications of integration.

We will define the (definite) integral as the area of a region under a graph. A basic question is how to define the said area.

Let us look at a justification for defining the area of the region enclosed by a circle of radius r. We assume that we know the area of a given triangle and we approximate the region enclosed by the given circle as follows. For an arbitrary n, consider the n equal inscribed and superscibed triangles as shown in Figure 1.

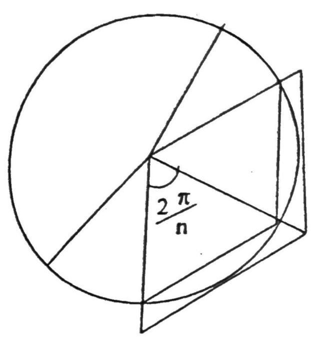

Figure 1

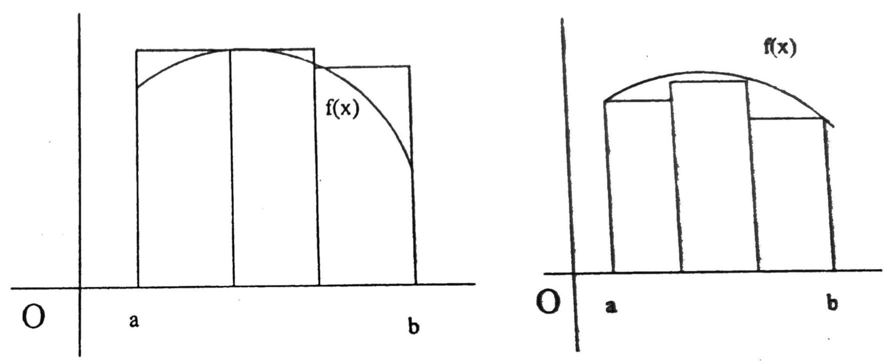

Figure 2

Observe that the total area of the inscribed triangles is $nr^2 sin(\pi/n)cos(\pi/n)$ and superscribed triangles is $nr^2 tan(\pi/n)$ (see Problem 1 of PP 16). Further, both $(nr^2 sin(\pi/n)cos(\pi/n))$ and $(nr^2 tan(\pi/n))$ converge to πr^2. **We will use this idea to define and evaluate the area of the region under a graph of a function.**

Suppose f is a non-negative bounded function defined on an interval $[a, b]$. We first subdivide the interval into a finite number of subintervals. Then we squeeze the region under the graph of f between the region covered by the inscribed and superscribed rectangles constructed over the subintervals as shown in Figure 2. **If** the total areas of the inscribed and superscribed rectangles come closer to a common value as we make the partition of $[a, b]$ finer and finer then the area of the region under the graph of f can be defined as this common value and f is said to be integrable.

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Let us define whatever has been explained above formally.

The Riemann Integral

Let $[a, b]$ be a given interval. A *partition* P of $[a, b]$ is a finite set of points $x_0, x_1, x_2, \ldots, x_n$ such that $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ and we write $P = \{x_0, x_1, x_2, \ldots, x_n\}$.

If $P = \{x_0, x_1, x_2, \ldots, x_n\}$ is a partition of $[a, b]$ we denote $\Delta x_i = x_i - x_{i-1}$ for $1 \leq i \leq n$. Throughout this and the next two lectures, we assume that f is a bounded function on $[a, b]$. For the given partition P of $[a, b]$, we define

$$M_i = \sup\{f(x) : \, x_{i-1} \leq x \leq x_i\}, \qquad m_i = \inf\{f(x) : \, x_{i-1} \leq x \leq x_i\}.$$

$$U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i \;\; \textit{and} \;\; L(P, f) = \sum_{i=1}^{n} m_i \Delta x_i.$$

The numbers $U(P, f)$ and $L(P, f)$ are called the *upper and lower Riemann sums* for the partition P (see Figure 2).

Consider two partitions P_1 and P_2 of $[a, b]$ such that $P_1 \subset P_2$, i.e., the points which are in P_1 are also in P_2 and P_2 has some extra points. Intuitively, it is clear that $L(P_1, f) \leq L(P_2, f)$ and $U(P_2, f) \leq U(P_1, f)$. Moreover, intuitively, we feel that if we add more and more points to a partition then the upper sums get smaller and the lower sums get larger. Let us formally prove the above statements, which we guessed intuitively.

Definition 16.1. A partition P_2 of $[a, b]$ is said to be finer than a partition P_1 if $P_2 \supset P_1$. In this case we say that P_2 is a *refinement* of P_1.

Theorem 16.1. *Let P_2 be a refinement of P_1 then $L(P_1, f) \leq L(P_2, f)$ and $U(P_2, f) \leq U(P_1, f)$.*

Proof (*). We first assume that P_2 contains just one more point than P_1. Let this extra point be $x^\star$. Suppose $x_{i-1} < x^\star < x_i$, where x_{i-1} and x_i are consecutive points of P_1. Let

$$w_1 = \inf\{f(x) : \, x_{i-1} \leq x \leq x^\star\} \qquad \text{and} \qquad w_2 = \inf\{f(x) : \, x^\star \leq x \leq x_i\}.$$

Then $w_1 \geq m_i$ and $w_2 \geq m_i$ where $m_i = \inf\{f(x) : \, x_{i-1} \leq x \leq x_i\}$. Then

$$\begin{aligned} L(P_2, f) - L(P_1, f) &= w_1(x^\star - x_{i-1}) + w_2(x_i - x^\star) - m_i(x_i - x_{i-1}) \\ &= (w_1 - m_i)(x^\star - x_{i-1}) + (w_2 - m_i)(x_i - x^\star) \\ &\geq 0 \end{aligned}$$

If P_2 contains k more points then we repeat this process $k-$times. The other inequality is analogously proved. $\square$

Our aim is to make the upper sums as large as possible and the lower sums as small as possible by considering different partitions so that the "area" of the region under the graph which is to be defined is squeezed between the lower sums and the upper sums. One may think that we can start with a partition and then go on taking its refinements so that this aim can be achieved. But which partition to start with and which way to refine it are the natural questions. So, why not considering all the possible partitions to achieve our goal. In light of this, we define

$$\bar{\int_a^b} f(x)dx = \inf\{U(P, f) : P \text{ is a partition of } [a, b]\}$$

and

$$\underline{\int_a^b} f(x)dx = \sup\{L(P,f) : P \text{ is a partition of } [a,b]\}.$$

Note that, since f is bounded, there exist real numbers m and M such that $m \le f(x) \le M$, for all $x \in [a,b]$. Thus for every partition P of $[a,b]$,

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a).$$

Hence the sets $\{U(P,f) : P$ is a partition of $[a,b]\}$ and $\{L(P,f) : P$ is a partition of $[a,b]\}$ are bounded. Therefore, $\overline{\int_a^b} f dx$ and $\underline{\int_a^b} f(x)dx$ exist and are called the *upper and lower Riemann integrals* of f over $[a,b]$ respectively.

Definition 16.2. (i) A bounded function $f : [a,b] \to \mathbb{R}$ is said to be Riemann *integrable* or *integrable* (on $[a,b]$) if $\overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx$.

(ii) If f is integrable on $[a,b]$, then the common value $\overline{\int_a^b} f(x)dx$ $(= \underline{\int_a^b} f(x)dx)$ is called the Riemann integral of f and it is denoted by $\int_a^b f(x)dx$.

Examples 16.1. 1. Consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(\frac{1}{2}) = 1 \text{ and } \ f(x) = 0 \text{ for all } x \in [0,1] \backslash \{\frac{1}{2}\}.$$

Then f is integrable. We show this using the definition as follows. For any partition P of $[0,1]$, $L(P,f)$ is always 0 and hence the lower integral is 0. Let us evaluate the upper integral. Let $P = \{x_0, x_1, x_2, .., x_n\}$ be any partition of $[0,1]$ and $\frac{1}{2} \in [x_{i-1}, x_i]$ for some i. If $\frac{1}{2} \in (x_{i-1}, x_i)$ then

$$U(P,f) = M_i \triangle x_i = \triangle x_i \le \max\{\Delta x_j : 1 \le j \le n\} \le 2\max\{\Delta x_j : 1 \le j \le n\}.$$

If $\frac{1}{2} = x_{i-1}$, then

$$U(P,f) = M_{i-1}\triangle x_{i-1} + M_i \triangle x_i = \triangle x_{i-1} + \triangle x_i \le 2\max\{\Delta x_j : 1 \le j \le n\}.$$

Similarly, if $\frac{1}{2} = x_i$, then we can show that $U(P,f) \le 2\max\{\Delta x_j : 1 \le j \le n\}$. Since we can always choose a partition P such that $\max\{\Delta x_j : 1 \le j \le n\}$ is as small as possible, the upper integral, which is the infimum of $U(P,f)'s$, is 0. Hence, f is integrable and $\int_0^1 f(x)dx = 0$.

2. Not every bounded function is integrable. For example, consider the function f defined by

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Consider an interval $[a,b]$. For any partition P of $[a,b]$, $U(P,f) = b-a$ and $L(P,f) = 0$. Hence the upper integral of f is 1 and the lower integral is 0. Therefore f is not integrable over any interval $[a,b]$.

In general, determining whether a bounded function on $[a,b]$ is integrable, using the definition, is difficult. For the purpose of checking the integrability, we give a criterion for integrability, called Riemann criterion, which is analogous to the Cauchy criterion for the convergence of a sequence.

Let us define some concepts and results before presenting the criterion.

Definition 16.3. Given two partition P_1 and P_2, the partition $P_1 \cup P_2 = P$ is called their common refinement.

The geometric interpretation suggests that the lower integral is less than or equal to the upper integral. So the next result is also anticipated.

Theorem 16.2. *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then $\overline{\int_a^b} f(x)dx \geq \underline{\int_a^b} f(x)dx$.*

Proof (*). Let P_1, P_2 be two partitions of $[a,b]$ and let P be their common refinement. Then by Theorem 16.1,

$$L(P_1, f) \leq L(P, f) \leq U(P, f) \leq U(P_2, f).$$

Thus for any two partitions P_1 and P_2, we have $L(P_1, f) \leq U(P_2, f)$. Fix P_2 and take supremum over all P_1. Then $\underline{\int_a^b} f(x)dx \leq U(P_2, f)$. Now take infimum over all P_2 to get the desired result. □

In the following result we present the Reimann criterion (a necessary and sufficient condition for the existence of the integral of a bounded function).

Theorem 16.3. (Riemann's criterion for integrability). *Let $f : [a,b] \to \mathbb{R}$ be a bounded function. Then f is integrable if and only if for every $\epsilon > 0$ there exists a partition P such that*

$$U(P, f) - L(P, f) < \epsilon. \tag{1}$$

Proof (*). Suppose that condition (1) holds. Let $\epsilon > 0$ and P satisfy (1). Then

$$L(P, f) \leq \underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx \leq U(P, f).$$

Therefore, (1) implies that $\overline{\int_a^b} f(x)dx - \underline{\int_a^b} f(x)dx < \epsilon$. Since ϵ is arbitrary, $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$. Ths shows that f is integrable.

Conversely, suppose f is integrable and $\epsilon > 0$. Then there exist partitions P_1 and P_2 such that

$$L(P_1, f) > \int_a^b f(x)dx - \epsilon/2 \quad \text{and} \quad U(P_2, f) < \int_a^b f(x)dx + \epsilon/2.$$

Let P be the common refinement of P_1 and P_2. Then

$$\int_a^b f(x)dx - \frac{\epsilon}{2} < L(P_1, f) \leq L(P, f) \leq U(P, f) \leq U(P_2, f) < \int_a^b f(x)dx + \frac{\epsilon}{2}.$$

Therefore $U(P, f) - L(P, f) < \epsilon$. □