Lecture 19: Improper integrals

We defined $\int_a^b f(t)dt$ under the conditions that f is defined and bounded on the bounded interval $[a,b]$. In this lecture, we will extend the theory of integration to bounded functions defined on unbounded intervals and also to unbounded functions defined on bounded or unbounded intervals.

Improper integral of the first kind

Let $a \in \mathbb{R}$. Suppose f is (Riemann) integrable on $[a,x]$ for all $x > a$, i.e., $\int_a^x f(t)dt$ exists for all $x > a$. If $\lim\limits_{x\to\infty} \int_a^x f(t)dt = L$ for some $L \in \mathbb{R}$, then we say that the improper integral (of the first kind) $\int_a^\infty f(t)dt$ converges to L and we write $\int_a^\infty f(t)dt = L$. Otherwise, we say that the improper integral $\int_a^\infty f(t)dt$ diverges.

Observe that the definition of convergence of improper integrals is similar to the one given for series. For example, $\int_a^x f(t)dt,\ x > a$ is analogous to the partial sum of a series.

Example 19.1 (i) The improper integral $\int_1^\infty \frac{1}{t^2}dt$ converges, because, $\int_1^x \frac{1}{t^2}dt = \int_1^x \frac{d}{dt}(-\frac{1}{t})dt = 1 - \frac{1}{x} \to 1$ as $x \to \infty$.

On the other hand, $\int_1^\infty \frac{1}{t}dt$ diverges because $\lim_{x\to\infty} \int_1^x \frac{1}{t}dt = \lim_{x\to\infty} \log x$. In fact, one can show that $\int_1^\infty \frac{1}{t^p}dt$ converges to $\frac{1}{p-1}$ for $p > 1$ and diverges for $p \leq 1$.

(ii) Note that $\int_0^x e^{-t}dt = 1 - e^{-x} \to 1$ as $x \to \infty$. Hence $\int_0^\infty \frac{1}{e^t}dt$ converges and $\int_0^\infty \frac{1}{e^t}dt = 1$.

(iii) Consider $\int_0^\infty te^{-t^2}dt$. We will use substitution in this example as follows. Note that

$$\int_0^x te^{-t^2}dt = \frac{1}{2}\int_0^{x^2} e^{-s}ds = \frac{1}{2}(1 - e^{-x^2}) \to \frac{1}{2}\ as\ x \to \infty.$$

Hence, it follows from (ii) that $\int_0^\infty te^{-t^2}dt = \frac{1}{2} = \frac{1}{2}\int_0^\infty e^{-s}ds.$

(iv) The integral $\int_0^\infty \sin t dt$ diverges, because, $\int_0^x \sin t dt = 1 - \cos x$ which does not converge as $n \to \infty$ (why?).

We now derive some convergence tests for improper integrals. These tests are similar to those used for series.

We first present a necessary and sufficient condition which is analogous to the result: *If $a_n \geq 0$ for all n, then $\sum_{n-=1}^\infty a_n$ converges if and only if the sequence of partial sum (S_n) is bounded.*

Theorem 19.1. *Suppose f is integrable on $[a,x]$ for all $x > a$ and $f(t) \geq 0$ for all $t > a$. Then $\int_a^\infty f(t)dt$ converges if and only if*

$$\textit{there exists } M > 0 \textit{ such that } \int_a^x f(t)dt \leq M \textit{ for all } x \geq a. \tag{1}$$

Proof (*). Let $\int_a^\infty f(t)dt$ converge to L for some $L \in \mathbb{R}$. Then there exists $N \in \mathbb{N}$ such that $\int_a^x f(t)dt \leq L+1$ for all $x \geq N$ which proves (1). The proof of the converse is similar to that of Theorem 3.2. $\square$

Example 19.2. It follows from Theorem 19.1 that the integral $\int_0^\infty |\sin x|dx$ diverges.

One uses Theorem 19.1 to prove the following theorem which is analogous to the comparison test for series.

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In the following two results we assume that f and g are integrable on $[a,x]$ for all $x>a$.

Theorem 19.2 (Comparison test). *Suppose* $0 \leq f(t) \leq g(t)$ *for all* $t>a$. *If* $\int_a^\infty g(t)dt$ *converges, then* $\int_a^\infty f(t)dt$ *converges.*

Examples 19.2. 1. The improper integral $\int_1^\infty \frac{cos^2 t}{t^2}dt$ converges, because $0 \leq \frac{cos^2 t}{t^2} \leq \frac{1}{t^2}$.

2. The improper integral $\int_1^\infty \frac{2+sint}{t}dt$ diverges, because $\frac{2+sint}{t} > 0$ and $\frac{2+sint}{t} \geq \frac{1}{t}$ for all $t>1$.

The proof of the following result is similar to that of Theorem 13.4.

Theorem 19.3 (Limit Comparison Test(LCT)). *Suppose* $f(t) \geq 0$ *and* $g(t) > 0$ *for all* $x>a$. *If* $\lim\limits_{t\to\infty} \frac{f(t)}{g(t)} = c$ *where* $c \neq 0$, *then both the integrals* $\int_a^\infty f(t)dt$ *and* $\int_a^\infty g(t)dt$ *converge or both diverge. In case* $c=0$, *then the convergence of* $\int_a^\infty g(t)dt$ *implies the convergence of* $\int_a^\infty f(t)dt$.

Examples 19.3. 1. For $p\in\mathbb{R}$, $\int_1^\infty e^{-t}t^p dt$ converges by the LCT because $\frac{e^{-t}t^p}{t^{-2}} \to 0$ as $t\to\infty$.

2. The integral $\int_1^\infty \sin\frac{1}{t}dt$ diverges whereas $\int_1^\infty (1-t\sin\frac{1}{t})dt$ converges by LCT (see Example 13.5). Note that in these two integrals the integrands are positive.

Theorems 19.2 and 19.3 are applicable for improper integrals of only non-negative functions. To deal with the functions which are not non-negative, the following result will be useful.

Theorem 19.4. *Suppose* $\int_a^x f(t)dt$ *exists for all* $x>a$. *If* $\int_a^\infty |f(t)|dt$ *converges then* $\int_a^\infty f(t)dt$ *converges i.e., every absolutely convergent improper integral is convergent.*

Proof. Suppose $\int_a^\infty |f(t)|dt$ converges. Since $0 \leq f(x)+|f(x)| \leq 2|f(x)|$ for all $x>a$, by the comparison test $\int_a^\infty (f(x)+|f(x)|)dx$ converges. This implies that $\int_a^\infty (f(x)+|f(x)|-|f(x)|)dx$ converges, i.e., $\int_a^\infty f(t)dt$ converges. □

The converse of Theorem 19.4 need not be true (see Exercise 19.2).

Example 19.4. Let $p>1$. Since $\left|\frac{\sin x}{x^p}\right| \leq \frac{1}{x^p}$ for $x\in[1,\infty)$, it follows from Example 19.1 and Theorem 19.4 that $\int_1^\infty \frac{\sin x}{x^p}dt$ converges.

The following result, known as **Dirichlet test**, is very useful (see PP 12 for the Dirichlet test for series).

Theorem 19.5 : *Let* $f,g:[a,\infty)\to\mathbb{R}$ *be such that*

(i) f *is decreasing and* $f(t)\to 0$ *as* $t\to\infty$;

(ii) g *is continuous and there exists* M *such that* $\left|\int_a^x g(t)dt\right| \leq M$ *for all* $x>a$.

Then $\int_a^\infty f(t)g(t)dt$ *converges.*

Proof of Theorem 19.5 is oulined in Problem 10 of PP 19.

Example 19.5. By Dirichlet's test, the integrals $\int_\pi^\infty \frac{sint}{t}dt$, $\int_\pi^\infty \frac{sint}{\sqrt{t}}dt$ and $\int_1^\infty \frac{\cos t}{t}dt$ are convergent.

Improper integrals of the form $\int_{-\infty}^b f(t)dt$ are defined similarly.

We say that $\int_{-\infty}^\infty f(t)dt$ is convergent if both $\int_{-\infty}^c f(t)dt$ and $\int_c^\infty f(t)dt$ are convergent for some element c in $\mathbb{R}$ and if it converges then we define $\int_{-\infty}^\infty f(t)dt = \int_{-\infty}^c f(t)dt + \int_c^\infty f(t)dt$ (see Example 19.7)

Improper integral of the second kind

Let $a, b \in \mathbb{R}$ and $a < b$. Suppose $\int_x^b f(t)dt$ exists for all x such that $a < x \leq b$ (the function f could be unbounded on $(a, b]$). If $\lim\limits_{x \to a^+} \int_x^b f(t)dt = M$ for some $M \in \mathbb{R}$, then we say that the improper integral (of the second kind) $\int_a^b f(t)dt$ converges to M and we write $\int_a^b f(t)dt = M$.

Example 19.6. The improper integral $\int_0^1 \frac{1}{t^p}dt$ converges for $p < 1$ and diverges for $p \geq 1$.

The comparison test and the limit comparison test for improper integral of the second kind are analogous to those of the first kind. If an improper integral is a combination of both first and second kind then one defines the convergence similar to that of the improper integral of the kind $\int_{-\infty}^{\infty} f(t)dt$ (see Exercise 19.1).

Example 19.7. 1. We show that $\int_{-\infty}^{\infty} \frac{1}{1+t^2}dt = \pi$. We first show that $\int_0^{\infty} \frac{1}{1+t^2}dt = \frac{\pi}{2}$. Note that for $x > 0$,

$$\int_0^x \frac{1}{1+t^2}dt = \left[\tan^{-1}(x)\right]_0^x = \tan^{-1}(x) - \tan^{-1}(0) = \tan^{-1}(x).$$

Since $\tan^{-1}(x) \to \frac{\pi}{2}$ as $x \to \infty$, $\int_0^{\infty} \frac{1}{1+t^2}dt = \frac{\pi}{2}$. Further, $\int_{-\infty}^{0} \frac{1}{1+t^2}dt = \frac{\pi}{2}$. As per our defnition,

$$\int_{-\infty}^{\infty} \frac{1}{1+t^2}dt = \int_{-\infty}^{0} \frac{1}{1+t^2}dt + \int_0^{\infty} \frac{1}{1+t^2}dt = \pi.$$

2. Let $f(x) = x$ and $g(x) = \sin x$ for all $x \in \mathbb{R}$. Then $\lim_{x \to \infty} \int_x^x f(t)dt = 0$ and $\lim_{x \to \infty} \int_x^x g(t)dt = 0$. But the integrals, $\int_0^{\infty} f(t)dt$, $\int_0^{\infty} g(t)dt$, $\int_{-\infty}^{0} f(t)dt$ and $\int_{-\infty}^{0} g(t)dt$ do not converge. Therefore, as per our definition, $\int_{\infty}^{\infty} f(t)dt$ and $\int_{\infty}^{\infty} g(t)dt$ do not converge.

3. Should we consider the integral $\int_0^1 \frac{\sin t}{t}dt$ as a proper or improper integral? The question arises because the integrand is bounded but it is not defined at 0. Since $\frac{\sin x}{x} > 0$ on $(0, 1]$ and $\frac{\sin x}{x} \leq 1$ (resp. $\lim_{x \to 0^+} \frac{sinx/x}{1} \to 1$), by the comparison test (resp. LCT), $\int_0^1 \frac{\sin t}{t}dt$ converges. Moreover, if we define $f(x) = \frac{\sin x}{x}$ for $x \neq 0$ and $f(0) = 0$. Then f is a continuous function and hence integrable. Further, $\int_0^1 \frac{\sin t}{t}dt = \int_0^1 f(t)dt$ which can be seen as follows. Let $0 < c < 1$. Then,

$$\int_0^1 f(t)dt = \int_0^c f(t)dt + \int_c^1 \frac{\sin t}{t}dt. \tag{2}$$

Since $f(t) \leq 1$ on $[0, c]$, $\lim_{c \to 0^+} \int_0^c f(t)dt = 0$. Hence if we take $c \to 0^+$ in the left hand and right hand sides of the equation (2), we get that $\int_0^1 f(t)dt = \int_0^1 \frac{\sin t}{t}dt$.

In the following exercise, we illustrate that an integrand can behave differently in different domains.

Exercise 19.1. Determine the values of p for which $\int\limits_0^{\infty} f(x)dx$ converges where $f(x) = \frac{1-e^{-x}}{x^p}$.

Solution. Let $I_1 = \int\limits_0^1 f(x)dx$ and $I_2 = \int\limits_1^{\infty} f(x)dx$. We have to determine the values of p for which both the integrals I_1 and I_2 converge. Now one has to see how the function $f(x)$ behaves in the respective intervals and apply the LCT. Near 0 we use $e^{-x} \approx 1 - x$. Since $\lim\limits_{x \to 0^+} \frac{1-e^{-x}}{x} = 1$, $\lim\limits_{x \to 0^+} \frac{f(x)}{1/x^{p-1}} = 1$. Hence by the LCT we see that I_1 is convergent if and only if $p - 1 < 1$, *i.e.*, $p < 2$.

As $x \to \infty$, we use $e^{-x} \approx 1$. Note that $\lim_{x\to\infty} \frac{f(x)}{1/x^p} = 1$. Hence by the LCT, I_2 is convergent if and only if $p > 1$. Therefore $\int\limits_0^\infty f(x)dx$ converges if and only if $1 < p < 2$.

Exercise 19.2. Let $0 < p \leq 1$. Show that $\int_1^\infty \frac{\sin x}{x^p} dx$ converges but not absolutely.

Solution. By Dirichlet's Test, the given integral converges for $0 < p \leq 1$. We claim that $\int\limits_1^\infty |\frac{\sin x}{x^p}|dx$ does not converge for $0 < p \leq 1$. Since, $|\sin x| \geq \sin^2 x$, we see that, for $x \in [1, \infty)$,

$$\left|\frac{\sin x}{x^p}\right| \geq \frac{\sin^2 x}{x^p} = \frac{1 - \cos 2x}{2x^p}.$$

Note that by Dirichlet's Test, $\int\limits_1^\infty \frac{\cos 2x}{2x^p} dx$ converges for all $p > 0$. But $\int\limits_1^\infty \frac{1}{2x^p}$ diverges for $p \leq 1$. Hence $\int_1^\infty \frac{1-\cos 2x}{2x^p} dx$ diverges for $0 < p \leq 1$. Therefore by the comparison test, $\int\limits_1^\infty |\frac{\sin x}{x^p}|dx$ diverges for $0 < p \leq 1$.