Lecture 2: Convergence of Sequences

In Lecture 1, some properties of $\mathbb{R}$ were established. In this lecture, we introduce the concept of convergence of a sequence of real numbers. We will see in the subsequent lectures that this concept is useful in

(i) establishing more properties of $\mathbb{R}$ which are needed;

(ii) studying continuity, differentiation and integration of functions;

(iii) approximating irrational numbers and also in finding numerical solutions to some equations.

Sequences

Intuitively, a sequence of real numbers can be thought of as a list of real numbers $x_1, x_2, \ldots$. In effect, we are associating a real number x_n for every $n \in \mathbb{N}$. Therefore, in a formal sense, a sequence of real numbers can be defined as a function from $\mathbb{N}$ to $\mathbb{R}$. Henceforth, the sequence $f : \mathbb{N} \to \mathbb{R}$ defined by $f(n) = x_n$ for all $n \in \mathbb{N}$, will be denoted by either $(x_1, x_2, x_3, \ldots)$ or (x_n). We call x_n the nth term of the sequence (x_n).

We list below some examples of sequences:

1. $(n) = (1, 2, 3, \ldots)$
2. $(\frac{1}{n}) = (1, \frac{1}{2}, \frac{1}{3}, \ldots)$
3. $(\frac{(-1)^n}{n}) = (-1, \frac{1}{2}, \frac{-1}{3}, \ldots)$
4. $(1 - \frac{1}{n}) = (0, 1 - \frac{1}{2}, 1 - \frac{1}{3}, \ldots)$
5. $((-1)^n) = (-1, +1, -1, +1, \ldots)$

Convergence of a Sequence

Before giving the formal definition of convergence of a sequence, let us take a look at the behaviour of the sequences in the above examples. In the case of the sequences, $(\frac{1}{n})$, $(1 - 1/n)$ and $((-1)^n/n)$, the nth term of each sequence appears to be *approaching* a fixed number as n gets larger. On the contrary, the nth term of the sequence (n) increases as n increases. In the case of the sequence $((-1)^n)$, the nth term oscillates between -1 and 1, accordingly as n is odd or even.

Let us focus our attention on those sequences (x_n), where the nth term x_n approaches a fixed number as n increases. To express the statement "x_n approaches a fixed number" in a formal sense, we need the concept of neighbourhood. For $a \in \mathbb{R}$ and $\epsilon > 0$, the $\epsilon-$neighbourhood of a is the open interval $(a - \epsilon, a + \epsilon) = \{x \in \mathbb{R} : a - \epsilon < x < a + \epsilon\}$. On visualizing geometrically, it appears that if the terms of the sequence (x_n) come eventually inside *every* $\epsilon-$neighbourhood of x_0 then x_n approaches x_0 as n increases. This geometric visualization leads to the following formal definition of convergence of a sequence of real numbers.

Definition 2.1. We say that a sequence (x_n) converges to a real number x_0 if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $x_n \in (x_0 - \epsilon, x_0 + \epsilon)$ whenever $n \geq N$.

Note that in Definition 2.1, the condition "$x_n \in (x_0 - \epsilon, x_0 + \epsilon)$ whenever $n \geq N$" can be replaced by "$|x_n - x_0| < \epsilon$ for all $n \geq N$". Observe that the value of N depends on the choice of ϵ.

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In general, as the value of ϵ becomes smaller, the value of N becomes larger. This is illustrated in Example 2.1.

Example 2.1. 1. Consider the sequence $(\frac{1}{\sqrt{n}})$. We will show that this sequence converges to 0, using Definition 2.1. Let ϵ be any given positive real number. We have to find $N \in \mathbb{N}$ such that $|\frac{1}{\sqrt{n}} - 0| < \epsilon$ for all $n \geq N$. Note that the inequality $\frac{1}{\sqrt{n}} < \epsilon$ is true for all $n > \frac{1}{\epsilon^2}$. Hence choose any $N \in \mathbb{N}$ such that $N > \frac{1}{\epsilon^2}$. Then for every $n \geq N$, we have $n > \frac{1}{\epsilon^2}$. Thus $|\frac{1}{\sqrt{n}} - 0| < \epsilon$ for all $n \geq N$. This shows that $(\frac{1}{\sqrt{n}})$ converges to 0.

2. Let $x_n = 8$ for all n (such a sequence is called constant sequence), and $\epsilon > 0$ be given. Then $|x_n - 8| < \epsilon$ for all $n \geq 1$. Hence $x_n \to 8$. In this case, we take $N = 1$ for any given ϵ.

We now see that a sequence cannot converge to more than one real number. To prove this, assume that (x_n) converges to x_0 and y_0, and $x_0 < y_0$. Let $\epsilon = (y_0 - x_0)/4$. By Definition 2.1, there exists $N_1 \in \mathbb{N}$ such that $x_n \in (x_0 - \epsilon, x_0 + \epsilon)$ whenever $n \geq N_1$ and there exists $N_2 \in \mathbb{N}$ such that $x_n \in (y_0 - \epsilon, y_0 + \epsilon)$ whenever $n \geq N_2$. Thus $x_n \in (x_0 - \epsilon, x_0 + \epsilon) \cap (y_0 - \epsilon, y_0 + \epsilon)$ whenever $n \geq \max\{N_1, N_2\}$. This is not possible, because, $(x_0 - \epsilon, x_0 + \epsilon) \cap (y_0 - \epsilon, y_0 + \epsilon) = \emptyset$.

If (x_n) converges to some x_0 then we say that (x_n) *converges* or (x_n) is a *convergent sequence*. We call x_0 the *limit* of the sequence (x_n) and we write $\lim\limits_{n\to\infty} x_n = x_0$ or simply $x_n \to x_0$. A sequence which does not converge is called a *divergent sequence*. For example, the sequence $((-1)^n)$ is a divergent sequence (see Problem 3 of PP2).

Elementary results about convergent sequences

We can show, as shown in Example 2.1, that $\frac{1}{n} \to 0$, $\frac{1}{n^2} \to 0$ and $1 + \frac{1}{10^n} \to 1$. In each of these examples, from the behaviour of the elements of the sequence, it was possible to guess the limit and then verify it (using the definition). However, in most cases, it will not be possible to follow this method. In some cases, it might be possible to guess the limit, but verifying the convergence from the definition will be difficult. Clearly, there is a need for more tools to address the issues of convergence of sequences and the computation of their limits. In this context, the *Limit Theorem, Sandwich Theorem and Ratio Test* (*for sequences*) are presented below.

Theorem 2.1 (*Limit Theorem*). *Suppose $x_n \to x$ and $y_n \to y$. Then*

(i) $x_n + y_n \to x + y$

(ii) $x_n y_n \to xy$

(iii) $\frac{x_n}{y_n} \to \frac{x}{y}$ if $y \neq 0$ and $y_n \neq 0$ for all n.

Example 2.2. 1. Let (x_n) be the sequence where $x_n = \frac{1}{1^2+1} + \frac{1}{2^2+2} + \cdots + \frac{1}{n^2+n}$ for every $n \in \mathbb{N}$. We can rewrite x_n as $x_n = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$. Since $\frac{1}{n} \to 0$, using Theorem 2.1 (i), we see that (x_n) converges and the limit of (x_n) is 1.

2. Let (x_n) be the sequence where $x_n = \frac{n^3+8n^2+2n}{3n^3+2}$. Here, $x_n = \frac{1+\frac{8}{n}+\frac{2}{n^2}}{3+\frac{2}{n^3}}$. Use Theorem 2.1 and verify that $x_n \to \frac{1}{3}$.

Theorem 2.2 (*Sandwich Theorem*). *Let $(x_n), (y_n)$ and (z_n) be sequences, where $x_n \leq y_n \leq z_n$ for all n and both (x_n) and (z_n) converge to x_0. Then $y_n \to x_0$.*

Proof: Let $\epsilon > 0$ be given. Since $x_n \to x_0$ and $z_n \to x_0$, there exist $N_1 \in \mathbb{N}$ and $N_2 \in \mathbb{N}$ such that

$$x_n \in (x_0 - \epsilon, x_0 + \epsilon) \text{ for all } n \geq N_1$$

and

$$z_n \in (x_0 - \epsilon, x_0 + \epsilon) \text{ for all } n \geq N_2.$$

Choose $N = max\{N_1, N_2\}$. Since $x_n \leq y_n \leq z_n$, we have

$$y_n \in (x_0 - \epsilon, x_0 + \epsilon) \text{ for all } n \geq N.$$

This proves that $y_n \to x_0$. $\square$

Example 2.3. 1. Since $\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$, by the sandwich theorem $\frac{\sin n}{n} \to 0$.

2. If $x_n = \frac{n^2}{n^3+n+1} + \frac{n^2}{n^3+n+2} + \cdots + \frac{n^2}{n^3+2n}, n \in \mathbb{N}$, then $\frac{n.n^2}{n^3+2n} \leq x_n \leq \frac{n.n^2}{n^3+n+1}$. Thus $x_n \to 1$.

3. If $x \in \mathbb{R}$ and $0 < x < 1$, then we show that $x^n \to 0$ as follows. Since $0 < x < 1$, there exists $a > 0$, such that that $x = \frac{1}{1+a}$. By Bernoulli's inequality (see Problem 2 of PP1), $0 < x^n = \frac{1}{(1+a)^n} \leq \frac{1}{1+na} < \frac{1}{na}$. From the sandwich theorem, we get $x^n \to 0$.

4. Let $x \in \mathbb{R}$ and $x > 0$. We show that $x^{\frac{1}{n}} \to 1$. Suppose $x > 1$ and $x^{\frac{1}{n}} = 1 + d_n$ for some $d_n > 0$. By Bernoulli's inequality, $x = (1 + d_n)^n > 1 + nd_n > nd_n$ which implies that $0 < d_n < \frac{x}{n}$ for all $n \in \mathbb{N}$. By the sandwich theorem $d_n \to 0$ and hence $x^{\frac{1}{n}} \to 1$. If $0 < x < 1$, let $y = \frac{1}{x}$ so that $y^{\frac{1}{n}} \to 1$ and hence $x^{\frac{1}{n}} \to 1$.

5. We show that $n^{\frac{1}{n}} \to 1$. Let $n^{\frac{1}{n}} = 1 + k_n$ for some $k_n > 0$ when $n > 1$. Hence $n = (1 + k_n)^n > 1$ for $n > 1$. If $n > 1$, then by the binomial theorem, $n \geq 1 + \frac{1}{2}n(n-1)k_n^2$. Therefore, for $n > 1$, $n - 1 \geq \frac{1}{2}n(n-1)k_n^2$ and hence $k_n \leq \sqrt{\frac{2}{n}}$. By the sandwich theorem $k_n \to 0$ and therefore $n^{\frac{1}{n}} \to 1$.

We need the following definition for stating the ratio test for sequence.

Definition 2.2. We say that (x_n) diverges to ∞ and write $x_n \to \infty$ if, for every $M > 0$, there exists $N \in \mathbb{N}$ such that $x_n > M$ whenever $n \geq N$. We say that (x_n) diverges to $-\infty$ and write $x_n \to -\infty$ if, for every $M > 0$, there exists $N \in \mathbb{N}$ such that $x_n < -M$ whenever $n \geq N$.

Example 2.4. Let $x_n > 0$ for all $n \in \mathbb{N}$. Then $x_n \to 0$ if and only if $\frac{1}{x_n} \to \infty$ (see Problem 13 in PP2). Hence if $r > 1$, then $r^n \to \infty$. Similarly, for any $p > 0$, $n^p \to \infty$.

Theorem 2.3 *(Ratio test for sequence).* *Let $x_n > 0$ for all n and $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda$ for some $\lambda \geq 0$. Then*

(i) if $\lambda < 1$ then $x_n \to 0$;

(ii) if $\lambda > 1$ then $x_n \to \infty$.

Proof. (i) Since $\lambda < 1$, choose r such that $\lambda < r < 1$. As $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lambda$, there exists $N \in \mathbb{N}$ such that $\frac{x_{n+1}}{x_n} < r$ for all $n \geq N$. Hence, for all $n \geq N$,

$$0 < x_{n+1} < rx_n < r^2 x_{n-1} < \cdots < r^{n-N+1} x_N = \frac{x_N}{r^N} r^{n+1}.$$

As N is a fixed number, $C = \frac{x_N}{r^N}$ is a constant. Further, $0 < x_{n+1} < Cr^{n+1}$ for all $n \geq N$ and $r^n \to 0$ (as $0 < r < 1$). Using the sandwich theorem, we see that $x_n \to 0$.

(ii) Let $y_n = \frac{1}{x_n}$ for all $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} \frac{y_{n+1}}{y_n} = \frac{1}{\lambda} < 1$. By part (i), $y_n \to 0$. Hence $x_n \to \infty$.

Example 2.4. 1. Let $x_n = \frac{n}{2^n}, n \in \mathbb{N}$. Since $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{1}{2}$, by the ratio test, $x_n \to 0$.

2. Let us slightly change the preceding example. For every $n \in \mathbb{N}$, let $x_n = \frac{n^p}{q^n}$ for some $q > 1$ and $p > 0$. Apply the ratio test and show that $x_n \to 0$. Notice that both (n^p) and (q^n) diverge to ∞. The fact that $x_n \to 0$ reveals that (q^n) diverges to ∞ “faster” than (n^p).

3. Let $x_n = ny^{n-1}$ for some $y \in (0, 1)$. Since $\lim\limits_{n\to\infty} \frac{x_{n+1}}{x_n} = y$, by the ratio test, $x_n \to 0$.

4. In Theorem 2.3, if $\lambda = 1$ then we cannot conclude either the convergence or divergence of the given sequence. For example, consider the sequences $(n), (\frac{1}{n})$ and $(2 + \frac{1}{n})$. Observe that in each of these cases $\lambda = 1$. Further investigation is necessary if $\lambda = 1$.