Lecture 7: Rolle's Theorem, Mean Value Theorem

In Lecture 6, when the differentiation was introduced, it was stated that information about a given function can be extracted by looking at its derivative. This indicates that there must be a result which relates a function and its derivative. The mean value theorem (in short, MVT) is such a result and the MVT is one of the fundamental theorems in differential calculus. In this lecture, MVT will be derived as a consequence of Rolle's theorem which itself is a special case of the MVT.

Rolle's Theorem

Rolle's theorem is related to the classical maxima and minima problems in calculus as we will see. If a function is continuous on $[a, b]$ then Theorem 5.3 guaranties that it has points of maximum and minimum on $[a, b]$. But the result does not identify such points. We will see that if the function is differentiable, then the derivative helps in identifying such points.

Definition 7.1. Let I be an interval and $f : I \to \mathbb{R}$. A point $x_0 \in I$ is called a point of local maximum of f if there is $\delta > 0$ such that $f(x) \leq f(x_0)$ whenever $x \in I \cap (x_0 - \delta, x_0 + \delta)$. Similarly, a point of local minimum is defined.

Theorem 7.1 (A necessary condition). *Let $f : [a, b] \to \mathbb{R}$. Suppose $x_0 \in (a, b)$ is a point of local maximum or local minimum. If f is differentiable at x_0 then $f'(x_0) = 0$.*

Proof: Suppose $x_0 \in (a, b)$ is a point of local maximum. Find (x_n) and (y_n) such that $x_n, y_n \in (a, b)$, $x_n < x_0 < y_n$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = x_0$. Since x_0 is a point of local maximum, $f(x_n) - f(x_0) \leq 0$ and $f(y_n) - f(x_0) \leq 0$ for all $n > N$ for some $N \in \mathbb{N}$. Hence

$$f'(x_0) = \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} \geq 0.$$

Similarly,

$$f'(x_0) = \lim_{n\to\infty} \frac{f(y_n) - f(x_0)}{y_n - x_0} \leq 0.$$

Therefore $f'(x_0) = 0$. □

Remark 7.1. 1. Theorem 7.1 is not valid if x_0 is a or b. For example, if we consider the function $f : [0, 1] \to \mathbb{R}$ such that $f(x) = x$ then 1 is the point of maximum but $f'(1) = 1$.

2. The condition $f'(x_0) = 0$ stated in Theorem 7.1 is a necessary condition but it is not sufficient. That is, if $f'(x_0) = 0$ for some $x_0 \in (a, b)$, then it is not necessary that x_0 is point of local maximum or minimum. For example, consider $f(x) = x^3$ where $x \in [-1, 1]$. Then $0 \in (-1, 1)$ and $f'(0) = 0$. But x_0 is neither a point of local maximum or minimum.

3. Theorem 7.1 and the preceding remark imply that if $f'(x_0) = 0$ for some $x_0 \in (a, b)$, then x_0 is a candidate for local maximum or minimum. Hence, the statement of Theorem 7.1 is considered as the first derivative test for points of local maximum and minimum.

4. If $x_0 \in (a, b)$ is point of local maximum or minimum of f, then f need not be differentiable at x_0. For example, consider the functions $f(x) = 1 - |x|$ and $g(x) = |x|$ where $x \in [-1, 1]$.

The following theorem is an application of Theorem 7.1.

Theorem 7.2 (Rolle's Thorem). *Let f be continuous on $[a, b]$ and differentiable on (a, b). Suppose $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.*

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Proof: If f is a constant function on $[a, b]$ then, it follows from the definition of the derivative that $f'(c) = 0$ for all $c \in [a, b]$. Suppose there exists $x \in (a, b)$ such that $f(x) > f(a)$. (A similar argument can be given if $f(x) < f(a)$). Then by Theorem 5.3 there exists $c \in (a, b)$ such that c is a point of maximum. Hence by Theorem 7.1, we have $f'(c) = 0$. □

Application 7.1. Rolle's theorem can be used together with the IVT to determine the number of solutions of some equations. Three examples are presented here and some more examples can be found in PP7.

1. Consider the equation $x^{13} + 7x^3 - 5 = 0$. To determine the number of solutions of this equations, let $f(x) = x^{13} + 7x^3 - 5$. Then $f(0) < 0$ and $f(1) > 0$. By the IVT there is at least one positive root of $f(x)$. Since that $f(x) < 0$ for $x < 0$, $f(x)$ does not have a negative root. If there are two distinct positive roots of $f(x)$, then by Rolle's theorem there is some $x_0 > 0$ such that $f'(x_0) = 0$ which is not true. Therefore $f(x) = 0$ has exactly one real solution.

2. Consider the equation $f(x) = 0$ where $f(x) = x^{18} + e^{-x} + 5x^2 - 2\cos x$. Since $f''(x) > 0$ for all $x \in \mathbb{R}$, by Rolle's theorem, $f'(x)$ has at most one real root. Again by Rolle's theorem, $f(x)$ can have at most two real roots. Note that $f(0) < 0$, $f(2) > 0$ and $f(-2) > 0$. Hence, by the IVT $f(x) = 0$ has at least two real solutions. Hence $f(x) = 0$ has exactly two distinct real solutions.

3 (*Uniqueness of the n-th root*). Let $\alpha \in [0, \infty)$ and $n \in \mathbb{N}$. It was shown, using the IVT, in Lecture 6 that the equation $x^n = \alpha$ has a solution in $[0, \infty)$. We use Rolle's theorem to show that the equation $x^n = \alpha$ has exactly one solution in $[0, \infty)$. Let $f(x) = x^n - \alpha$ for $x \in [0, \infty)$. Then $f'(x) > 0$ for all $(0, \infty)$. Therefore by Rolle's theorem, $f(x) = 0$ cannot have more than one real solution in $[0, \infty)$.

Mean value theorem

A geometric interpretation of Rolle's theorem can be given as follows. Consider the graph of a function f which is continuous on $[a, b]$ and differentiable on (a, b). If the values of f at the end points a and b are equal then there is some $c \in (a, b)$ such that the graph has a horizontal tangent at $(c, f(c))$. It is natural to ask the following question. If the values of f at the end points a and b are not the same, is it true that there is some $c \in (a, b)$ such that the tangent line at $(c, f(c))$ is parallel to the line connecting the endpoints of the graph? The MVT answers in the affirmative.

Theorem 7.3 (MVT). *Let f be continuous on $[a, b]$ and differentiable on (a, b). Then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$.*

Proof: As MVT is derived as a consequence of Rolle's theorem, a new function g needs to be constructed out of f so that $g(a) = g(b)$. Let $g(x) = f(x) - h(x)$, where

$$h(x) = f(a) + \frac{f(b) - f(a)}{b - a}\,(x - a)$$

which is the equation of the chord joining $(a, f(a))$ and $(b, f(b))$. Observe that $g(a) = g(b) = 0$ and g satisfies the rest of the conditions of Rolle's theorem. Hence by Rolle's theorem, there exists $c \in (a, b)$ such that $g'(c) = 0$. This implies that $f'(c) - \frac{f(b)-f(a)}{b-a} = 0$ which proves the result. □

It is clear from the statement of the MVT that this result relates f and f'.

Application 7.2. The MVT is quite useful in extracting information about a given function f, from the available information about f'. This is illustrated with some examples below.

1. Let I be an interval and $f : I \to \mathbb{R}$ be such that $f'(x) = 0$ for all $x \in I$. Then f is a constant function on I.

To verify this, fix $x_0 \in I$ and choose any $x \in I$ such that $x \neq x_0$. Then by the MVT (applied for f on $[x, x_0]$ if $x < x_0$ or on $[x_0, x]$ if $x_0 < x$), there exists c between x and x_0 such that $f(x) - f(x_0) = f'(c)(x - x_0)$. Since $f'(c) = 0$, $f(x) = f(x_0)$. This shows that f is constant. $\square$

We make two remarks here.

(a) If f is a constant function on I, then it follows immediately from the definition of the derivative that $f'(x) = 0$ for all $x \in I$. The converse which is stated in the proceeding application, can perhaps be derived only from the MVT and not directly from the definition of the derivative.

(b) If I is not an interval, then f may not be constant if $f'(x) = 0$ for all $x \in I$. For example, consider $f : (0,1) \cup (2,3) \to \mathbb{R}$ which is defined by $f(x) = 0$ if $x \in (0,1)$ and $f(x) = 1$ if $x \in (2,3)$. Observe that $f'(x) = 0$ for all $x \in (0,1) \cup (2,3)$ but f is not constant.

2. Let I be an interval and $f : I \to \mathbb{R}$.

(i) If $f'(x) > 0$ for all $x \in I$. Then f is strictly increasing on I. If $f'(x) < 0$ for all $x \in I$. Then f is strictly decreasing on I.

(ii) If $f'(x) \neq 0$ for all $x \in I$, then f is one-one (i.e, $f(x) \neq f(y)$ whenever $x \neq y$).

(iii) If $f'(x) \geq 0$ for all $x \in I$ then f is increasing (i.e, $f(y) \geq f(x)$ if $x \geq y$). If $f'(x) \leq 0$ for all $x \in I$ then f is decreasing.

We verify the first statement. The other two statements can be verified similarly. To verify (i), let $x, y \in I$ be such that $x < y$. Then by the MVT, there exists $c \in (x, y)$ such that $f(y) - f(x) = f'(c)(y - x)$. Since $f'(c) > 0$, $f(y) > f(x)$. This shows that f is strictly increasing on I. $\square$

Note that f can be one-one and f' can also be 0 at some point. For example, if $f(x) = x^3$, $x \in \mathbb{R}$ then f is one-one and $f'(0) = 0$.

3. Let $f : [a, b] \to [a, b]$ be differentiable and $|f'(x)| \neq 1$ for all $x \in [a, b]$. Then f has a unique fixed point in $[a, b]$.

This is verified as follows. It is shown in Application 6.1 that f has a fixed point in $[a, b]$. Suppose f has two distinct fixed points x_1 and x_2 in $[a, b]$. Then $f(x_1) = x_1$ and $f(x_2) = x_2$. By the MVT, there exists c between x_1 and x_2 such that $f(x_1) - f(x_2) = f'(c)(x_1 - x_2)$. Therefore,

$$|x_1 - x_2| = |f(x_1) - f(x_2)| = |f'(c)||x_1 - x_2| \neq |x_1 - x_2|$$

which is a contradiction. $\square$

Application 7.3. Several inequalities can be obtained using the MVT. We present two examples below and some more examples are given in PP7.

1. For all $x, y \in \mathbb{R}$, $|\sin x - \sin y| \leq |x - y|$.

To see this, let $x, y \in \mathbb{R}$. By the MVT, $\sin x - \sin y = \cos c\ (x - y)$ for some c between x and y. Hence $|\sin x - \sin y| \leq |x - y|$.

2. For all $x \in \mathbb{R}$, $e^x \geq 1 + x$.

To verify the inequality, let $x > 0$. By the MVT there exists $c \in (0, x)$ such that $e^x - e^0 = e^c(x - 0)$. This implies that $e^x \geq 1 + x$. If $x < 0$, apply the MVT over the interval $[x, 0]$. $\square$