Practice Problems 11: Fixed point iteration method and Newton's method

1. Let $g : \mathbb{R} \to \mathbb{R}$ be differentiable and $\alpha \in \mathbb{R}$ be such that $|g'(x)| \le \alpha < 1$ for all $x \in \mathbb{R}$.

   (a) Show that the Picard sequence for g converges to a fixed point of g for any starting value $x_0 \in \mathbb{R}$.

   (b) Show that g has a unique fixed point.

2. Let $x_0 \in \mathbb{R}$. Using the fixed point iteration method generate a sequence of approximate solutions of the equation $x - \frac{1}{2}\sin x = 1$ for the starting value x_0.

3. Let $g : [0,1] \to [0,1]$ be defined by $g(x) = \frac{1}{1+x^2}$. Let (x_n) be the Picard sequence for g with the initial value $x_0 = 1$. Show that (x_n) converges.

4. Let $f(x) = e^{-\frac{1}{x^2}}$ if $x \ne 0$ and $f(0) = 0$. Suppose that $0 < x_0 < 1$ and (x_n) be the Newton sequence for f and x_0. Show that (x_n) converges.

5. Let $f(x) = 3x^{\frac{1}{3}}$. Let $x_0 > 0$ and (x_n) be the Newton sequence for f and x_0. Show that (x_n) oscillates and is unbounded.

6. Let $f : [-10, 10] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sqrt{x-1} & \text{if } x \ge 1 \\ -\sqrt{1-x} & \text{if } x < 1. \end{cases}$$

   Let $x_0 \ne 1$ and (x_n) be the Newton sequence for f and x_0. Show that $x_n = x_0$ if n is even and $x_n = 2 - x_0$ if n is odd.

7. Let $f : [a,b] \to \mathbb{R}$ be differentiable and $f'(x) \ne 0$ for all $x \in [a,b]$. Define F by $F(x) = x - \frac{f(x)}{f'(x)}$ for all $x \in [a,b]$. Let $F(x) \in [a,b]$ for all $x \in [a,b]$.

   (a) Suppose that f'' exists and for all $x \in [a,b]$ and

$$\left| \frac{f(x)f''(x)}{[f'(x)]^2} \right| \le \alpha < 1$$

   for some $\alpha \in \mathbb{R}$. Show that the Newton sequence (x_n) for f converges for any initial value $x_0 \in [a,b]$.

   (b) Let $f(x) = (x-1)^2$ and $x_0 \in [0,2]$. Show that the Newton sequence for f and x_0 converges to 1.

   (c) Let $f(x) = x^2 - 7$ and $x_0 \in [2,7]$. Show that the Newton sequence for f and x_0 converges to $\sqrt{7}$.

8. **(*)** Let $f : [a,b] \to [a,b]$ be continuous and ℓ be a fixed point of f. Suppose that f is differentiable on (a,b) and $|f'(x)| < 1$ for all $x \in (a,b)$. Let $x_0 \in [a,b]$ and $x_{n+1} = f(x_n)$ for $n = 0, 1, 2, \ldots$.

   (a) Show that f has a unique fixed point.

   (b) Show that $|x_{n+1} - \ell| \le |x_n - \ell|$ for all $n \in \mathbb{N}$.

   (c) If (x_{n_k}) is a subsequence of (x_n), show that $|x_{n_{k+1}} - \ell| \le |x_{n_k+1} - \ell| \le |x_{n_k} - \ell|$ for all $k \in \mathbb{N}$.

   (d) If a subsequence (x_{n_k}) of (x_n) converges to some x_0, show that $x_0 = \ell$.

   (e) Show that $x_n \to \ell$.

   (f) Show that for $f(x) = \frac{x^2}{2}$, $a = 0$ and $b = 1$ the sequence (x_n) converges.

Please write to psraj@iitk.ac.in if any typos/mistakes are found in this set of practice problems/solutions/hints.

Practice Problems 11: Hints/Solutions

1. Take $\mathbb{R}$ in place of $[a,b]$ and repeat the proof of Theorem 11.1.

2. Write $x = g(x)$ where $g(x) = 1 + \frac{1}{2}\sin x$ and note that $|g'(x)| \leq \frac{1}{2} < 1$ for all $x \in \mathbb{R}$. By Problem 1, the sequence (x_n) defined by $x_{n+1} = g(x_n)$ converges to a fixed point of g. Since a fixed point of g is a solution to the equation $x - \frac{1}{2}\sin x = 1$, the elements $x_n's$ are approximate solutions.

3. Observe that $g : [0,1] \to [0,1]$ and $|g'(x)| = \frac{2x}{(1+x^2)^2}$ achieves its maximum at $x = \frac{1}{\sqrt{3}}$ on $[0,1]$. Therefore $|g'(x)| \leq \frac{9}{8\sqrt{3}} < 1$ for all $x \in [0,1]$. Hence by Theorem 11.1, the sequence (x_n) converges.

4. For f, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3}{2}$. Then (x_n) is decreasing and bounded below.

5. In this case, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = -2x_n$. Therefore (x_n) is unbounded.

6. For given f, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = 2 - x_n$. This implies the answer.

7. (a) Observe that $F'(x) = \left|\frac{f(x)f''(x)}{[f'(x)]^2}\right|$. Apply Theorem 11.1.

   (b) Note that $F(x) = x - \frac{f(x)}{f'(x)} = \frac{1}{2}(x+1)$ and $F : [0,2] \to [0,2]$. Moreover $|F'(x)| = \left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| = \frac{1}{2} = \alpha < 1$ for all $x \in [0,2]$. So, the problem follows from (a) and Remark 11.2.

   (c) The function $F(x) = x - \frac{f(x)}{f'(x)} = \frac{x}{2} + \frac{7}{2x}$. It is shown in Problem 5 of PP3 that the sequence (x_n) defined by $x_{n+1} = \frac{1}{2}\left(x_n + \frac{7}{x_n}\right)$, converges.

   The problem can also be solved using (a) as follows. By finding the maximum and minimum values of the function $F(x)$ on $[2,7]$ or otherwise, verify that $F : [2,7] \to [2,7]$. Again, by finding the maximum and minimum values of the function $F'(x)$ or otherwise, verify that $|F'(x)| = \left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| \leq \frac{3}{7}$ for all $x \in [2,7]$. Therefore the problem follows from (a) and Remark 11.2.

8. (a) See the second part of the proof of Theorem 11.1.

   (b) By the mean value theorem $|x_{n+1} - \ell| = |f(x_n) - f(\ell)| < |x_n - \ell|$.

   (c) This follows from (b) and the definition of subsequence.

   (d) Suppose $x_{n_k} \to x_0$ and $x_0 \neq \ell$. Then $|x_{n_{k+1}} - \ell| \to |x_0 - \ell|$ and $|x_{n_k} - \ell| \to |x_0 - \ell|$. It follows from (c) that $|x_{n_k+1} - \ell| \to |x_0 - \ell|$; i.e., $|f(x_{n_k}) - f(\ell)| \to |x_0 - \ell| = |f(x_0) - f(\ell)|$. But by the mean value theorem $|f(x_0) - f(\ell)| < |x_0 - \ell|$ which is a contradiction.

   (e) Follows from (d), the Bolzano-Weierstrass Theorem and Problem 11 of PP3.

   (f) It is easily seen that $f : [0,1] \to [0,1]$ and $|f'(x)| < 1$ for all $x \in (0,1)$. So by (e), (x_n) converges.