Practice Problems 12: Convergence of a series, Leibniz test

1. Show that every sequence is a sequence of partial sums of a series.

2. Show that $\sum_{n=1}^{\infty}(a_n - a_{n+1})$ converges if and only if the sequence (a_n) converges. Verify the convergence/divergence of the following series:

   $(a)\ \sum_{n=1}^{\infty}\frac{4}{(4n-3)(4n+1)}$ $\quad(b)\ \sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}$ $\quad(c)\ \sum_{n=1}^{\infty}\frac{n}{(n^2+1)^2-n^2}$ $\quad(d)\ \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}+\sqrt{n-1}}$.

3. Show that the series $\frac{1}{2}+\frac{1}{3}+\frac{2}{3}+\frac{1}{4}+\frac{2}{4}+\frac{3}{4}+\frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5}+\frac{1}{6}+\frac{2}{6}+\cdots$ diverges.

4. Let (S_n) denote the sequence of partial sums of the series $\sum_{n=1}^{\infty}a_n$. Suppose $\sum_{n=1}^{\infty}S_n$ converges. Show that $\sum_{n=1}^{\infty}a_n$ converges.

5. Consider the sequence 0.2, 0.22, 0.222, 0.2222, ... By writing this sequence as a sequence of partial sums of a series, find the limit of this sequence.

6. In each of the following cases, discuss the convergence of the series $\sum_{n=1}^{\infty}a_n$ where a_n equals:

   (a) $\sin\left(\frac{(-1)^n}{n^p}\right), p > 0$ $\quad$ (b) $(-1)^n\frac{(\ln n)^3}{n}$ $\quad$ (c) $\frac{\cos(\pi n)\ln n}{n}$

7. Show that the series $\sum_{n=1}^{\infty}a_n$ converges if and only if for every $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|\sum_{i=m}^{n}a_i| < \epsilon$ for all $m, n \in \mathbb{N}$ satisfying $n \geq m \geq N$.

8. Let $\sum_{n=1}^{\infty}a_n$ be a convergent series. Consider $\sum_{n=1}^{\infty}b_n$ and $\sum_{n=1}^{\infty}c_n$ where $b_n = \max\{a_n, 0\}$ and $c_n = \min\{a_n, 0\}$ (i.e., series of positive terms and series of negative terms of $\sum_{n=1}^{\infty}a_n$).

   (a) If $\sum_{n=1}^{\infty}|a_n|$ converges then show that both $\sum_{n=1}^{\infty}b_n$ and $\sum_{n=1}^{\infty}c_n$ converge.
   (b) If $\sum_{n=1}^{\infty}|a_n|$ diverges then show that both $\sum_{n=1}^{\infty}b_n$ and $\sum_{n=1}^{\infty}c_n$ diverge.

9. Let $\sum_{n=1}^{\infty}a_n$ be a convergent series and $\sum_{n=1}^{\infty}b_n$ is obtained by grouping finite number of terms of $\sum_{n=1}^{\infty}a_n$ such as $(a_1 + a_2 + \cdots + a_{m_1}) + (a_{m_1+1} + a_{m_1+2} + \cdots + a_{m_2}) + \cdots$ for some $m_1, m_2, \ldots$ (Here $b_1 = a_1 + a_2 + \cdots + a_{m_1},\ b_2 = a_{m_1+1} + a_{m_1+2} + \cdots + a_{m_2}$ and so on). Show that $\sum_{n=1}^{\infty}b_n$ converges and has the same limit as $\sum_{n=1}^{\infty}a_n$. What happens if $\sum_{n=1}^{\infty}a_n$ diverges ?

10. Let $a_n \geq 0$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty}a_n$ be a convergent series. Suppose $\sum_{n=1}^{\infty}b_n$ is obtained by rearranging the terms of $\sum_{n=1}^{\infty}a_n$ *(i.e., the terms of* $\sum_{n=1}^{\infty}b_n$ *are same as those of* $\sum_{n=1}^{\infty}a_n$ *but they occur in different order).* Show that $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ converge to the same limit.

11. Consider the series $\sum_{n=1}^{\infty}a_n$ where $a_n = \frac{(-1)^{n+1}}{n}$. Show that the series
$$(1-\frac{1}{2})-\frac{1}{4}+(\frac{1}{3}-\frac{1}{6})-\frac{1}{8}+(\frac{1}{5}-\frac{1}{10})-\frac{1}{12}+\cdots,$$
which is obtained from $\sum_{n=1}^{\infty}a_n$ by rearranging and grouping, is $\frac{1}{2}\sum_{n=1}^{\infty}a_n$.

12. **(*)** Let (a_n) be a decreasing sequence, $a_n \geq 0$ and $\lim\limits_{n\to\infty}a_n = 0$. For each $n \in \mathbb{N}$, let $b_n = \frac{a_1+a_2+\cdots+a_n}{n}$. Show that $\sum_{n=1}^{\infty}(-1)^n b_n$ converges

13. **(*)** Consider the series $\sum_{n=1}^{\infty}a_n$ where $a_n = \frac{(-1)^{n+1}}{n}$. Let $\alpha \in \mathbb{R}$; for example take $\alpha = 2013$.

    (a) Show that there exists a smallest odd positive integer N_1 such that $1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{N_1} > 2013$. Further show that $1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{N_1}-\frac{1}{2} \leq 2013$.

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(b) Show that there exists a smallest odd positive integer $N_2 > N_1$ such that

$$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{N_1} - \frac{1}{2} + \frac{1}{N_1 + 2} + \cdots + \frac{1}{N_2} > 2013.$$

Further show that $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{N_1} - \frac{1}{2} + \frac{1}{N_1+2} + \cdots + \frac{1}{N_2} - \frac{1}{4} \leq 2013$.

(c) Show that $0 \leq \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{N_1} - \frac{1}{2} + \frac{1}{N_1+2} + ... + \frac{1}{N_2}\right) - 2013 \leq \frac{1}{N_2}$ and $0 \leq 2013 - \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{N_1} - \frac{1}{2} + \frac{1}{N_1+2} + \cdots + \frac{1}{N_2} - \frac{1}{4}\right) \leq \frac{1}{4}$.

(d) Following (b), consider the series of rearrangement

$1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{N_1} - \frac{1}{2} + \frac{1}{N_1+2} + \cdots + \frac{1}{N_2} - \frac{1}{4} + \frac{1}{N_2+2} + \cdots + \frac{1}{N_3} - \frac{1}{6} + \cdots$.

Show that

$\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{N_1} - \frac{1}{2} + \frac{1}{N_1+2} + \cdots + \frac{1}{N_2} - \frac{1}{4} + \frac{1}{N_2+2} + \cdots + \frac{1}{N_3}\right) - 2013 \leq \frac{1}{N_3}$.

Further, for any j such that $N_2 + 2 \leq N_2 + 2j \leq N_3 - 2$, show that

$2013 - \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{N_1} - \frac{1}{2} + \frac{1}{N_1+2} + \cdots + \frac{1}{N_2} - \frac{1}{4} + \frac{1}{N_2+2} + \cdots + \frac{1}{N_2+2j}\right) \leq \frac{1}{4}$.

(e) Show that the series of rearrangement given in (d) converges to 2013.

14. **(*)** Let (A_n) and (S_n) be the sequences of partial sums of the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{a_n}{n}$ respectively. If $\sum_{n=1}^{\infty} a_n$ is convergent or the sequence of partial sums (A_n) is bounded then show that

(a) $S_{n+1} = A_1(\frac{1}{1} - \frac{1}{2}) + A_2(\frac{1}{2} - \frac{1}{3}) + \cdots + A_n(\frac{1}{n} - \frac{1}{n+1}) + \frac{A_n}{n+1}$, for $n > 1$;

(b) the series $|A_1(\frac{1}{1} - \frac{1}{2})| + |A_2(\frac{1}{2} - \frac{1}{3})| + \cdots + |A_n(\frac{1}{n} - \frac{1}{n+1})| + \cdots$ converges;

(c) the series $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

15. (a) **(*) (Dirichlet test)** Let $\sum_{n=1}^{\infty} a_n$ be a series whose sequence of partial sums is bounded. Let (b_n) be a decreasing sequence which converges to 0. Show that $\sum_{n=1}^{\infty} a_n b_n$ converges. Observe that Leibniz test is a particular case of the Dirichlet test.

(b) **(*) (Abel's test)** Let $\sum_{n=1}^{\infty} a_n$ be a convergent series and (b_n) be a monotonic convergent sequence. Show that $\sum_{n=1}^{\infty} a_n b_n$ converges.

(c) Show that the series $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \cdots$ converges whereas the series $1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} + \frac{1}{8} - \cdots$ diverges.

(d) Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}(1 + \frac{1}{n})^n$, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos \frac{1}{n}$ and $\sum_{n=1}^{\infty} (-1)^n \frac{\tan^{-1} n}{\sqrt{n}}$ converge.

Practice Problems 11 : Hints/Solutions

1. Let (a_n) be the given sequence. Consider the series $a_1 + (a_2 - a_1) + (a_3 - a_2) + \cdots$.

2. Note that the sequence of partial sums of the series $\sum_{n=1}^{\infty}(a_n - a_{n+1})$ is $(a_1 - a_{n+1})$.

   The series given in (a),(b) and (c) converge. The series given in (c) diverges.

3. The nth term of the series does not converge to 0.

4. If $\sum_{n=1}^{\infty} S_n$ converges, then $S_n \to 0$. Apply the definition of the convergence of a series.

5. The sequence is $\frac{2}{10}, \frac{2}{10} + \frac{2}{10^2}, \frac{2}{10} + \frac{2}{10^2} + \frac{2}{10^3}, \ldots$ which is a sequence of partial sums of the series $\sum_{n=1}^{\infty} \frac{2}{10^n}$. The given sequence converges to $\frac{2}{9}$.

6. (a) Converges by Leibniz test: $\sin(\frac{(-1)^n}{n^p}) = (-1)^n \sin(\frac{1}{n^p})$.

   (b) Converges by Leibniz test: If $f(x) = \frac{(\ln x)^3}{x}$ then $f'(x) < 0$ for all $x > e^3$.

   (c) Converges by Leibniz test: $\cos(\pi n) = (-1)^n$.

7. Use the fact that the series $\sum_{n=1}^{\infty} a_n$ converges if and only if its sequence of partial sums (S_n) satisfies the Cauchy criterion.

8. (a) Observe that $2b_n = a_n + |a_n|$ and $2c_n = a_n - |a_n|$ for all $n \in \mathbb{N}$.

   (b) Observe that $|a_n| = 2b_n - a_n$ and $|a_n| = a_n - 2c_n$ for all $n \in \mathbb{N}$.

9. Let (S_n) and $(\overline{S_k})$ be the sequences of partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ respectively. Observe that $(\overline{S_k})$ is a subsequence of (S_n). For the next part, consider the series $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ and the grouping $(1-1) + (1-1) + (1-1) + \cdots$.

10. Let (S_n) and $(\overline{S_n})$ be the sequences of partial sums of $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ respectively. Note that both (S_n) and $(\overline{S_n})$ are increasing sequences. Suppose $S_n \to S$ for some S. Then $\overline{S_n} \leq S$ for all n. Therefore $\overline{S_n}$ converges. If $\overline{S_n} \to \overline{S}$ for some $\overline{S}$, then $\overline{S} \leq S$. For the proof of $S \leq \overline{S}$, interchange a_n and b_n.

11. Trivial.

12. Note that $b_{n+1} - b_n = \frac{1}{n+1}(a_1 + a_2 + \ldots + a_{n+1}) - \frac{1}{n}(a_1 + \ldots + a_n) = \frac{a_{n+1}}{n+1} - \frac{(a_1+\ldots+a_n)}{n(n+1)}$. Since (a_n) is decreasing, $a_1 + \ldots + a_n \geq na_n$. Therefore, $b_{n+1} - b_n \leq \frac{a_{n+1}-a_n}{n+1} \leq 0$. Hence (b_n) is decreasing. It follows from Problem 15 of PP2 that $b_n \to 0$. Apply the Leibniz test.

13. (a) Since the series $1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots$ diverges, the sequence of the partial sums is unbounded. Therefore there exists a smallest odd positive integer N_1 such that $1+\frac{1}{3}+\cdots+\frac{1}{N_1} > 2013$. If $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{N_1-2} + \frac{1}{N_1} - \frac{1}{2} > 2013$, then $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{N_1-2} > 2013$ as $\frac{1}{N_1} - \frac{1}{2} < 0$ which is a contradiction.

    (b) Similar to (a).

    (c) From (b), it follows that $1 + \frac{1}{3} + \frac{1}{5} + \ldots + \frac{1}{N_1} - \frac{1}{2} + \frac{1}{N_1+2} + \cdots + \frac{1}{N_2-2} \leq 2013$. That is $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{N_1} - \frac{1}{2} + \frac{1}{N_1+2} + \ldots + \frac{1}{N_2-2} + \frac{1}{N_2} \leq 2013 + \frac{1}{N_2}$. This implies the first inequality of (c).

    From the first inequality of (b), $1+\frac{1}{3}+\frac{1}{5}+\cdots+\frac{1}{N_1}-\frac{1}{2}+\frac{1}{N_1+2}+\cdots+\frac{1}{N_2}-\frac{1}{4} > 2013-\frac{1}{4}$. This implies the second inequality of (c).

    (d) The proof of the first part is similar to the proof of the first part of (c). The second part follows from the second part of (c).

    (e) Observe from (c) and (d) that the sequence of partial sums of the series of rearrangement converges to 2013.

14. (a) Use the fact that $a_n = A_n - A_{n-1}$.

    (b) Since (A_n) is a bounded sequence, let $|A_n| \leq M$ for all $n \in \mathbb{N}$ and for some M. Therefore $|A_1(\frac{1}{1} - \frac{1}{2})| + |A_2(\frac{1}{2} - \frac{1}{3})| + \cdots + |A_n(\frac{1}{n} - \frac{1}{n+1})| \leq M(1 - \frac{1}{n+1}) < M$.

(c) From (b), the sequence of partial sums of the series $A_1(\frac{1}{1}-\frac{1}{2})+A_2(\frac{1}{2}-\frac{1}{3})+\cdots$ converges. Therefore (S_n) converges.

15. (a) Compare the Dirichlet Test with Practice Problems 12. Repeat the steps (a)-(c) given in the problem mentioned above by taking b_n in place of $\frac{1}{n}$.

(b) Compare Abel's Test with Practice Problems 12. Repeat the steps (a)-(c) given in the problem mentioned above by taking b_n in place of $\frac{1}{n}$. In Abel's test (b_n) could be increasing. However, the proofs of the steps (a)-(c) go through.

(c) Convergence of the first series follows from the Dirichlet test. For the divergence of the second series, consider the sequence of partial sums (for instance, $S_3 \geq \frac{1}{3}$, $S_6 \geq \frac{1}{3}+\frac{1}{6}$, $S_9 \geq \frac{1}{3}+\frac{1}{6}+\frac{1}{9}$,...).

(d) Apply Abel's test.