Practice Problems 13: Comparison, Limit comparison and Cauchy condensation tests

1. Let $a_n \geq 0$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} a_n$ converges then show that

   (a) $\sum_{n=1}^{\infty} a_n^2$ converges (Is the converse true?);

   (b) $\sum_{n=1}^{\infty} \sqrt{a_n a_{n+1}}$ converges;

   (c) $\sum_{n=1}^{\infty} \frac{\sqrt{a_n}}{n}$ converges;

   (d) $\sum_{n=1}^{\infty} \frac{a_n+4^n}{a_n+5^n}$ converges using comparison or limit comparison test.

2. Let (a_n) be a sequence such that $a_n > 0$ for all n and $a_n \to \infty$. Show that $\sum_{n=1}^{\infty} \frac{1}{a_n^n}$ converges.

3. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Show that $\sum_{n=1}^{\infty} |a_n|$ diverges if $\sum_{n=1}^{\infty} a_n^2$ diverges.

4. Let $a_n > 0$ for all $n \in \mathbb{N}$. Show that the series $\sum_{n=1}^{\infty} \frac{a_1+a_2+\ \dots\ +a_n}{n}$ diverges.

5. (a) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge absolutely, show that $\sum_{n=1}^{\infty} a_n b_n$ converges absolutely.

   (b) If $\sum_{n=1}^{\infty} a_n$ converges absolutely and (b_n) is a bounded sequence then $\sum_{n=1}^{\infty} a_n b_n$ converges absolutely.

   (c) Give an example of a convergent series $\sum_{n=1}^{\infty} a_n$ and a bounded sequence (b_n) such that $\sum_{n=1}^{\infty} a_n b_n$ diverges.

6. Let $a_n, b_n \geq 0$ for all $n \in \mathbb{N}$. Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent. Show that $\sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}$ converges. Does the converse hold ?

7. Let $a_n, b_n \in \mathbb{R}$ for all n and $\sum_{n=1}^{\infty} a_n^2$ and $\sum_{n=1}^{\infty} b_n^2$ converge. Show that $\sum_{n=1}^{\infty} (a_n - b_n)^p$ converges for all $p \geq 2$.

8. Let $a_n \geq 0$. Show that both the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converge or diverge together.

9. Show that $\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{n+1}\right)$ diverges.

10. Let $a_n \geq 0$ for all n and $n^3 a_n^2 \to \ell$ for some $\ell > 0$. Show that $\sum_{n=1}^{\infty} \frac{a_n}{\sqrt{n}}$ converges.

11. Suppose $a_n > 0$ for all n and $\sum_{n=1}^{\infty} a_n$ converges. Show that the series $\sum_{n=1}^{\infty} \left(1 - \frac{\sin a_n}{a_n}\right)$ converges.

12. Let $a_n \geq 0$ and $a_{n+1} \leq a_n$ for all n. Suppose $\sum_{n=1}^{\infty} a_n$ converges. Using the Cauchy condensation test, show that $n a_n \to 0$ as $n \to \infty$.

13. Consider the series $\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{1}{n}$ for $n = 1, 4, 9, 16, \dots$ and $a_n = \frac{1}{n^2}$ otherwise (i.e., if n is not a perfect square). Show that $\sum_{n=1}^{\infty} a_n$ converges but $n a_n \nrightarrow 0$.

14. Let (a_n) be a sequence of positive real numbers such that $a_{n+1} \leq a_n$ for all n and $\sum_{n=1}^{\infty} a_n$ converge. Show that $\sum_{n=1}^{\infty} n(a_n - a_{n+1})$ converges.

15. Show that $\sum_{n=4}^{\infty} \frac{1}{n(\ln n)(\ln(\ln n))}$ diverges.

Please write to psraj@iitk.ac.in if any typos/mistakes are found in this set of practice problems/solutions/hints.

16. In each of the following cases, discuss the convergence/divergence of the series $\sum_{n=2}^{\infty} a_n$ where a_n equals:

(a) $\frac{1}{(\ln n)^p}, (p > 0)$ (b) $\frac{\sin(\frac{1}{n})}{\sqrt{n}}$ (c) $\frac{2+n}{n^{7/4}\ln n}$ (d) $\frac{1}{n^2-\ln n}$ (e) e^{-n^2}

(f) $\frac{1}{n^{1+\frac{1}{n}}}$ (g) $\tan\frac{1}{n}$ (h) $1-\cos\frac{\pi}{n}$ (i) $(\ln n)\sin\frac{1}{n^2}$ (j) $\frac{\tan^{-1} n}{n\sqrt{n}}$

(k) $(n+2)(1-\cos\frac{1}{n})$ (ℓ) $\frac{3+\cos n}{e^n}$ (m) $\frac{2+\sin^3(n+1)}{2^n+n^2}$ (n) $\frac{\sqrt{n+1}-\sqrt{n}}{n}$

17. **(*)** Suppose that $a_n > 0$ for all n and $\sum_{n=1}^{\infty} a_n$ diverges. Let (S_n) be the sequence of partial sums of $\sum_{n=1}^{\infty} a_n$ and (A_n) be the sequence of partial sums of $\sum_{n=1}^{\infty} \frac{a_n}{S_n}$

   (a) Show that (A_n) does not satisfy the Cauchy criterion.

   (b) Show that there exists a sequence (b_n) such that $b_{n+1} \le b_n$ for all n, $b_n \to 0$ and $\sum_{n=1}^{\infty} b_n a_n$ also diverges.

Practice Problems 13: Hints/Solutions

1. (a) Since $a_n \to 0$, $a_n^2 \le a_n$ eventually. The converse is not true: Take $a_n = n^{-\frac{2}{3}}$.

   (b) Use the inequality $\sqrt{a_n a_{n+1}} \le \frac{1}{2}(a_n + a_{n+1})$.

   (c) Use $\sqrt{a_n \frac{1}{n^2}} \le \frac{1}{2}(a_n + \frac{1}{n^2})$.

   (d) Use $\frac{a_n+4^n}{a_n+5^n} \le \frac{a_n+4^n}{5^n} \le \left(\frac{1}{5}\right)^n + \left(\frac{4}{5}\right)^n$ or apply the LCT with $\left(\frac{4}{5}\right)^n$, i.e., find the $\lim_{n\to\infty} \frac{a_n+4^n}{a_n+5^n}\left(\frac{5}{4}\right)^n$.

2. Observe that $\frac{1}{a_n^n} < \frac{1}{2^n}$ eventually.

3. Since $a_n \to 0$, $a_n^2 \le |a_n|$ eventually.

4. Note that $\frac{a_1+a_2+\ \dots\ +a_n}{n} \ge \frac{a_1}{n}$.

5. (a) Since $b_n \to 0$, $|a_n b_n| \le |a_n|$ eventually. Use the comparison test.

   (b) Let $|b_n| \le M$ for some M. Then $|a_n b_n| \le M|a_n|$. Use the comparison test.

   (c) Consider $a_n = \frac{(-1)^n}{n}$ and $b_n = (-1)^n$.

6. Use the inequality $a_n^2 + b_n^2 \le (a_n + b_n)^2$. The converse is true, because $a_n \le \sqrt{a_n^2 + b_n^2}$.

7. It is sufficient to show that $\sum_{n=1}^{\infty}(a_n - b_n)^2$ converges because $|a_n - b_n|^p \le (a_n - b_n)^2$ eventually for $p > 2$. For convergence of $\sum_{n=1}^{\infty}(a_n - b_n)^2$, use the inequality $(a-b)^2 = 2a^2 + 2b^2 - (a+b)^2 \le 2a^2 + 2b^2$.

8. Suppose $\sum_{n=1}^{\infty} a_n$ converges. Since $0 \le \frac{a_n}{1+a_n} \le a_n$, $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges. Suppose $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges. Since $\frac{a_n}{1+a_n} \to 0$, $a_n \to 0$. Therefore $1 + a_n \le 2$ eventually. Hence $\frac{1}{2}a_n \le \frac{a_n}{1+a_n}$ eventually. By the comparison test $\sum_{n=1}^{\infty} a_n$ converges.

9. Use the LCT with $\frac{1}{n}$: $n\sin\left(\frac{n\pi}{n+1}\right) \to \pi$.

10. Use the LCT with $\frac{1}{n^2}$: $\frac{a_n}{\sqrt{n}}\frac{n^2}{1} = a_n n^{\frac{3}{2}} \to \sqrt{\ell} > 0$.

11. Use the LCT with a_n^2: $\frac{1}{a_n^2}\left(1 - \frac{\sin a_n}{a_n}\right) = \frac{a_n - \sin a_n}{a_n^3} \to \frac{1}{6}$.

12. By the Cauchy condensation test $\sum_{k=0}^{\infty} 2^k a_{2^k}$ converges. Therefore $2^k a_{2^k} \to 0$. For each $n \in \mathbb{N}$, choose $k \in \mathbb{N}$ such that $2^k \le n \le 2^{k+1}$. Then $na_n \le na_{2^k} \le 2^{k+1}a_{2^k} = 2 \cdot 2^k a_{2^k} \to 0$.

13. The series is $\frac{1}{1}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4}+\frac{1}{5^2}+\frac{1}{6^2}+\frac{1}{7^2}+\frac{1}{8^2}+\frac{1}{9}+\ldots$ The sequence of partial sums is bounded above by $(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots)+(1+\frac{1}{4}+\frac{1}{9}+\ldots)\le 2\sum_{n=1}^{\infty}\frac{1}{n^2}$ but $na_n=1$ when n is a perfect square.

14. The partial sum S_n of $\sum_{n=1}^{\infty} n(a_n-a_{n+1})$ is $a_1+a_2+\cdots+a_n-na_{n+1}$.

15. Use the Cauchy condensation test and the fact that $\ln 2<1$.

16. (a) Diverges (Use the LCT with $\frac{1}{n}$: $\frac{n}{(\ln n)^p}\to\infty$).

 (b) Converges (Use the LCT with $\frac{1}{n\sqrt{n}}$).

 (c) Diverges (Use the LCT with $\frac{1}{n^{3/4}\ln n}$).

 (d) Converges (Use the comparison test: $\frac{1}{n^2-\ln n}\le\frac{1}{n^2-n}\le\frac{1}{n(n-1)}$).

 (e) Converges (Use the comparison test: $\frac{1}{e^{n^2}}\le\frac{1}{n^2}$ as $e^x\ge x$).

 (f) Diverges (Use the LCT with $\frac{1}{n}$: $\frac{n}{n^{1+\frac{1}{n}}}\to 1$).

 (g) Diverges (Use the LCT with $\frac{1}{n}$: $\lim_{n\to\infty}\frac{\tan\frac{1}{n}}{\frac{1}{n}}=\lim_{n\to\infty}\frac{\sec^2(\frac{1}{n})(-\frac{1}{n^2})}{-\frac{1}{n^2}}=1$).

 (h) Converges (Use the LCT with $\frac{1}{n^2}$: $\frac{1-\cos\frac{\pi}{n}}{\frac{1}{n^2}}\to\frac{\pi^2}{2}$).

 (i) Converges (Use the LCT with $\frac{1}{n\sqrt{n}}$: $\frac{(\ln n)\sin\frac{1}{n^2}}{\frac{1}{n\sqrt{n}}}=\frac{\ln n}{\sqrt{n}}\frac{\sin\frac{1}{n^2}}{\frac{1}{n^2}}$).

 (j) Converges (Use the comparison test: $\frac{\tan^{-1}n}{n\sqrt{n}}\le\frac{\frac{\pi}{2}}{n\sqrt{n}}$).

 (k) Diverges because $(n+2)(1-\cos\frac{1}{n})\ge n(1-\cos\frac{1}{n})$ and $\sum_{n=1}^{\infty}n(1-\cos\frac{1}{n})$ diverges: $\frac{n(1-\cos\frac{1}{n})}{\frac{1}{n}}=\frac{1-\cos\frac{1}{n}}{\frac{1}{n^2}}\to\frac{1}{2}$.

 (ℓ) Converges (Use the comparison test: $0\le\frac{3+\cos n}{e^n}\le\frac{4}{e^n}=4(\frac{1}{e})^n$).

 (m) Converges because both $\sum_{n=1}^{\infty}\frac{2}{2^n+n^2}$ and $\sum_{n=1}^{\infty}\left|\frac{\sin^3(n+1)}{2^n+n^2}\right|$ converge.

 (n) Converges because $\frac{\sqrt{n+1}-\sqrt{n}}{n}=\frac{1}{n}\frac{1}{\sqrt{n+1}+\sqrt{n}}<\frac{1}{n^{\frac{3}{2}}}$.

17. (a) Note that, for any $p\in\mathbb{N}$, $|A_{n+p}-A_n|\ge\frac{a_{n+1}+a_{n+2}+\ \ldots\ +a_{n+p}}{S_{n+p}}=\frac{S_n+p-S_n}{S_{n+p}}\to 1$ as $p\to\infty$.

 (b) Take $b_n=\frac{1}{S_n}$.