# Practice Problems 16: Integration, Riemann's Criterion for integrability (Part I)

1. Prove the inequality $nr^2\sin(\pi/n)\cos(\pi/n) \leq A \leq r^2\tan(\pi/n)$ given in the lecture notes where $A$ is the area of the circle of radius $r$.

2. Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Suppose that there is a partition $P$ of $[a, b]$ such that $L(P, f) = U(P, f)$. Show that $f$ is a constant function.

3. Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $f(x) \geq 0$ for every $x \in [a, b]$. Show that $\underline{\int_a^b} f(x)dx \geq 0$ and $\overline{\int_a^b} f(x)dx \geq 0$. In addition, if $f$ is integrable, show that $\int_a^b f(x)dx \geq 0$.

4. In each of the following cases, evaluate the upper and lower integrals of $f$ and show that $f$ is integrable. Find the integral of $f$.

   (a) For $\alpha \in \mathbb{R}$, define $f : [a, b] \to \mathbb{R}$ by $f(x) = \alpha$ for every $x \in [a, b]$.

   (b) $f(x) = 0$ for $0 \leq x < \frac{1}{2}$, $f(\frac{1}{2}) = 10$ and $f(x) = 1$ for $\frac{1}{2} < x \leq 1$.

   (c) $f(x) = x$ for all $x \in [0, 1]$.

5. Let $f : [a, b] \to \mathbb{R}$ be integrable and $(P_n)$ be a sequence of partitions of $[a, b]$ such that $U(P_n, f) - L(P_n, f) \to 0$.

   (a) Show that $\lim_{n\to\infty} L(P_n, f) = \lim_{n\to\infty} U(P_n, f) = \int_a^b f(x)dx$.

   (b) Find $\int_0^1 xdx$ using $(a)$.

   (c) Find $\int_0^1 x^2dx$ using $(a)$

6. Let $f(x) = \frac{1}{x}$ for all $x \in [1, 2]$. Show that $f$ is integrable using Theorem 17.1.

7. Let $f, f_1$ and $f_2$ be bounded functions on $[0, 1]$ such that $f_1(x) \leq f(x) \leq f_2(x)$ for all $x \in [0, 1]$. Suppose that $f_1$ and $f_2$ are integrable and $\int_0^1 f_1(x)dx = \int_0^1 f_2(x)dx$, show that $f$ is integrable and find $\int_0^1 f(x)dx$.

8. Let $f : [0, 1] \to \mathbb{R}$ be such that $f(x) = x$ for $x$ rational and $f(x) = 0$ for $x$ irrational. Evaluate the upper and lower integrals of $f$ and show that $f$ is not integrable.

9. **(*)** Let $f : [0, 1] \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p, q \in \mathbb{N} \text{ and } \ p, q \text{ have no common factors} \\ 0 & \text{if } x \text{ is irrational or } x = 0 \end{cases}$$

   (a) For any $N \in \mathbb{N}$ consider the set

$$A_N = \left\{ x \in [0, 1] : x = \frac{p}{q} \text{ where } p, q \in \mathbb{N}, q \leq N \text{ and } p, q \text{ have no common factors} \right\}.$$

   Show that the set $A_N$ is finite.

   (b) For given $N \in \mathbb{N}$ and $\epsilon > 0$, show that there are disjoint intervals $[x_1, x_2], [x_3, x_4], ..., [x_{m-1}, x_m]$ such that $A_N \subseteq (x_1, x_2) \cup (x_3, x_4) \cup ... \cup (x_{m-1}, x_m)$ and $|x_1 - x_2| + |x_3 - x_4| + ... + |x_{m-1} - x_m| \leq \frac{\epsilon}{2}$.

   (c) Show that $f$ is integrable.

   (d) Find two integrable functions $g$ and $h$ on $[0, 1]$ such that $g \circ h$ ($g$ composition of $h$) is not integrable.

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Please write to psraj@iitk.ac.in if any typos/mistakes are found in this set of practice problems/solutions/hints.

## Practice Problems 16: Hints/Solutions

1. The area of the inscribed triangle given in Figure 1 in the notes is $2 \times \frac{1}{2} r \sin(\pi/n) r \cos(\pi/n)$. The area of the superscribed triangle is $2 \times \frac{1}{2}(r \tan(\pi/n))r$.

2. Observe that $U(P,f) - L(P,f) = \sum_{i=1}^{n}(M_i - m_i)\Delta x_i$ and $M_i - m_i \geq 0$ and $\Delta x_i > 0$.

3. Follows from the definitions.

4. (a) For any partition $P = \{x_0, x_1, ..., x_n\}$ of $[a,b]$, $m_i = M_i = \alpha$ for $i = 1, 2, ..., n$. and hence $U(P,f) = L(P,f) = \alpha(b-a)$. Therefore $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx = \alpha(b-a)$. This implies that $f$ is integrable and $\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \alpha(b-a)$.
   (b) Let $P = \{x_0, x_1, ..., x_n\}$ be any partition of $[0,1]$ such that $\frac{1}{2} \in (x_{i-1}, x_i)$ for some $1 \leq i \leq n$. Then $L(P,f) = 1 - x_i$ and $U(P,f) = 10\Delta x_i + (1 - x_i)$. Therefore $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx = \frac{1}{2}$. This implies that $f$ is integrable and $\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \frac{1}{2}$.
   (c) Let $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, \frac{n}{n}\right\}$. By definition $L(P_n, f) = \frac{(n-1)n}{2n^2}$ and $U(P_n, f) = \frac{n(n+1)}{2n^2}$. Therefore
   $\frac{1}{2} = \sup\{L(P_n, f) : n \in \mathbb{N}\} \leq \underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx \leq \inf\{U(P_n, f) : n \in \mathbb{N}\} = \frac{1}{2}$.
   Therefore $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx = \frac{1}{2}$ and $\int_a^b f(x)dx = \frac{1}{2}$.

5. (a) Follows from the fact that $L(P_n, f) \leq \int_a^b f(x)dx \leq U(P_n, f)$.
   (b) It is shown in Example 17.2 that $U(P_n, f) - L(P_n, f) \to 0$ if $P_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n}\}$. Hence by $(a)$, $\lim_{n\to\infty} U(P_n, f) = \int_a^b f(x)dx$. It follows from the solution of 4(c) that $U(P_n, f) = \frac{n(n+1)}{2n^2} \to \frac{1}{2}$.
   (c) Follow the argument involved in the solution of Problem 5(b). If $f(x) = x^2$, then $U(P_n, f) = \frac{1}{n}(\frac{1}{n^2} + \frac{2^2}{n^2} + ... + \frac{n^2}{n^2}) = \frac{1}{n}\frac{n(n+1)(2n+1)}{n^2 6} \to \frac{1}{3}$.

6. Let $P_n = \left\{1, 1 + \frac{1}{n}, 1 + \frac{2}{n}, ..., 1 + \frac{n-1}{n}, 1 + \frac{n}{n}\right\}$. Then $U(P_n, f) - L(P_n, f) = \frac{1}{2n} \to 0$.

7. For any partition $P$ of $[0,1]$, $L(P, f_1) \leq L(P, f)$ and $U(P,f) \leq U(P_2, f)$ which implies that $\int_0^1 f_1(x)dx \leq \underline{\int_0^1} f(x)dx \leq \overline{\int_0^1} f(x)dx \leq \overline{\int_0^1} f_2(x)dx = \int_0^1 f_2(x)dx = \int_0^1 f_1(x)dx$.

8. If $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, \frac{n}{n}\right\}$ then $\overline{\int_a^b} f(x)dx \leq \inf\{U(P_n, f) : n \in \mathbb{N}\} = \frac{1}{2}$ (see the solution of Problem 4(c)). If $P = \{x_0, x_1, x_2, ..., x_n\}$ be any partition of $[0,1]$, then
   $U(P,f) = \sum_{i=1}^{n} x_i(x_i - x_{i-1}) \geq \sum_{i=1}^{n} x_i^2 - \frac{1}{2}(\sum_{i=1}^{n}(x_i^2 + x_{i-1}^2)) = \frac{1}{2}(\sum_{i=1}^{n}(x_i^2 - x_{i-1}^2)) = \frac{1}{2}$
   which implies that $\overline{\int_a^b} f(x)dx \geq \frac{1}{2}$. Therefore $\overline{\int_a^b} f(x)dx = \frac{1}{2}$. It is clear that $\underline{\int_a^b} f(x)dx = 0$.

9. (a) It is clear that $A_N$ is finite.
   (b) Since the set $A_N$ is finite, this is possible.
   (c) See the argument involved in Example 17.1. Let $\epsilon > 0$. Choose $N$ such that $\frac{1}{N} < \frac{\epsilon}{2}$. Corresponding to this $N$, choose the partition $P = \{0, x_1, x_2, x_3, ..., x_n, 1\}$ of $[0,1]$ where $x_i's$ are as given in (b).
   Observe that if $x \in [x_2, x_3]$ or $[x_4, x_5]$ and $f(x) = \frac{1}{q}$ then $q \geq N$ and hence on these intervals $M_i - m_i \leq \frac{1}{N}$.

   Note that
   $U(P,f) - L(P,f) = \sum(M_i - m_i)\Delta x_i = (|x_1 - x_2| + |x_3 - x_4| + ... + |x_{m-1} - x_m|) + \frac{1}{N} < \epsilon$.
   This shows that $f$ is integrable.
   (d) Define $g(0) = 0$ and $g(x) = 1$ if $x \in (0,1]$. Take $h = f$ where $f$ is defined above.