Practice Problems 19: Improper Integrals

1. Show that $\int_1^\infty \frac{1}{t^p}dt$ converges to $\frac{1}{p-1}$ if $p > 1$ and it diverges to ∞ if $p \leq 1$.

2. Let $f : [a, \infty) \to \mathbb{R}$ be differentiable and f' be integrable on $[a, x]$ for all $x \geq a$. Show that $\int_a^\infty f'(t)dt$ converges if and only if $\lim_{t\to\infty} f(t)$ exists.

3. Find the limits of the following improper integrals.

   (a) $\int_0^{\pi/2} \ln t dt$ (b) $\int_0^1 \ln \frac{1}{t} dt$ (c) $\int_0^\infty e^{-t}dt$

   (d) $\int_0^\infty \frac{dt}{e^t+e^{-t}}dt$ (e) $\int_1^\infty p^t dt,\ 0 < p < 1$

4. ***(Cauchy Criterion)*** Let $f : [a, \infty) \to \mathbb{R}$ be integrable on $[a, x]$ for all $x \geq a$. Show that $\int_a^\infty f(t)dt$ converges if and only if for every $\epsilon > 0$ there exists $N \geq a$ such that $\left|\int_x^y f(t)dt\right| < \epsilon$ for every $x, y \geq N$.

5. Let $f : [0, \infty) \to \mathbb{R}$ be defined by $f(t) = \frac{(-1)^{n+1}}{n}$ when $t \in [n - 1, n)$, $n \in \mathbb{N}$. Show that $\int_0^\infty f(t)dt$ converges but not absolutely.

6. Let $f : [1, \infty) \to \mathbb{R}$ be defined by $f(n) = 1$ for all $n \in \mathbb{N}$ and $f(x) = 0$ if $x \in [1, \infty)\backslash\mathbb{N}$. Then show that

   (a) $\int_1^\infty f(t)dt$ converges but $\sum_{n=1}^\infty f(n)$ diverges.

   (b) $\int_1^\infty (f(t) - 1)dt$ diverges but $\sum_{n=1}^\infty (f(n) - 1)$ converges.

7. ***(Integral Test)*** Let $f : [1, \infty) \to \mathbb{R}$ be a non-negative decreasing function. Then show that

   (a) (μ_n) is decreasing and bounded below where $\mu_n = (\sum_{k=1}^n f(k)) - \int_1^n f(t)dt$.

   (b) either both $\sum_{n=1}^\infty f(n)$ and $\int_1^\infty f(t)dt$ converge or else both diverge.

8. (a) Let $f : [1, \infty) \to \mathbb{R}$ be such that $f(n) = 1$ for all $n \in \mathbb{N}$ and $f(t) = 0$ otherwise. Show that $\int_1^\infty f(t)dt$ converges but $f(t) \nrightarrow 0$ as $t \to \infty$.

   (b) Does there exist a continuous function $f : [1, \infty) \to \mathbb{R}$ such that $\int_1^\infty f(t)dt$ converges but $f(t) \nrightarrow 0$ as $n \to \infty$?

9. Determine the values of k for which the improper integral $\int_1^\infty \left[\frac{kt}{1+t^2} - \frac{1}{2t}\right] dt$ converges.

10. ***(Drichlet Test)*** Let $f, g : [a, \infty) \to \mathbb{R}$ be such that

    (a) f is continuous, decreasing and $f(t) \to 0$ as $t \to \infty$,

    (b) there exists M such that $\left|\int_a^x g(t)dt\right| \leq M$ for all $x > a$.

    Then $\int_a^\infty f(t)g(t)dt$ converges.

11. Determine the values of p for which the following improper integrals converge.

    (a) $\int_1^\infty \frac{\sin t}{t^p}dt$ (b) $\int_1^\infty \frac{\ln t}{t^p}dt$ (c) $\int_0^\infty \frac{t^{p-1}}{1+t}dt$

    (d) $\int_1^\infty t^p e^{-t}dt$ (e) $\int_0^1 \frac{1-\cos t}{t^p}dt$.

12. ***(Root Test)*** Let $f : [a, \infty) \to \mathbb{R}$ be such that f is integrable on $[a, x]$ for all $x > a$. Suppose $|f(t)|^{\frac{1}{t}} \to \ell$ as $t \to \infty$ for some $\ell \in \mathbb{R}$ or $\ell = \infty$. Then

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Please write to psraj@iitk.ac.in if any typos/mistakes are found in this set of practice problems/solutions/hints.

(a) if $\ell < 1$, then the integral $\int_a^\infty f(t)dt$ converges absolutely.

(b) if $\ell > 1$ and f is non-negative then the integral $\int_a^\infty f(t)dt$ diverges.

13. Determine the convegence/divergence of the following integrals.

(a) $\int\limits_0^1 \frac{\sqrt{t}}{e^{\sin t - 1}} dt.$ (b) $\int\limits_0^{\frac{\pi}{2}} \ln(\sin t)dt$ (c) $\int\limits_0^\infty \frac{1}{t^2+\sqrt{t}} dt$ (d) $\int\limits_0^1 \cos \frac{1}{t^2} dt.$

(e) $\int\limits_0^\infty \sin t^3 dt$ (f) $\int\limits_1^\infty \frac{\sin 2t}{\sqrt{t}} e^{\sin t} dt$ (g) $\int\limits_1^\infty t \sin t^4 dt$ (h) $\int\limits_0^{\frac{\pi}{4}} \frac{dt}{t - \sin t}.$

(i) $\int\limits_1^\infty \frac{1-5\sin 2t}{t^2+\sqrt{t}} dt$ (j) $\int\limits_0^1 \frac{e^{\frac{t}{2}}}{\sqrt{1-\cos t}} dt$ (k) $\int\limits_1^\infty \frac{t^t}{e^{2t}} dt$ (ℓ) $\int\limits_1^\infty \frac{e^t}{4^t} dt.$

14. ***(Gamma Function)*** Show that the following function Γ, called Gamma function, is well defined: $\Gamma : (0, \infty) \to \mathbb{R}$ given by $\Gamma(p) = \int_0^\infty e^{-t} t^{p-1} dt$.

Practice Problems 19: Hints/Solutions

1. If $p \neq 1$ then for $x \in [1, \infty)$, $\int_1^x \frac{1}{t^p} dt = \frac{x^{1-p}-1}{1-p}$. If $p = 1$, then for $x \in [1, \infty)$, $\int_1^x \frac{1}{t} dt = \ln x$.

2. By the FTC, $\int_a^x f'(t)dt = f(x) - f(a)$, for $x \in [a, \infty)$.

3. (a) $\lim_{x\to 0} \int_x^{\frac{\pi}{2}} \ln t dt = \lim_{x\to 0} [t \ln t - t]_x^{\frac{\pi}{2}} = \frac{\pi}{2}[\ln \frac{\pi}{2} - 1]$.

   (b) $\lim_{x\to 0} \int_x^1 \ln \frac{1}{t} dt = \lim_{x\to 0} [t - t \ln t]_x^1 = 1$.

   (c) $\lim_{x\to\infty} \int_0^x e^{-t} dt = \lim_{x\to\infty} [-e^{-t}]_0^x = \lim_{x\to\infty} [1 - e^{-x}] = 1$.

   (d) $\lim_{x\to\infty} \int_0^x \frac{e^t}{e^{2t}+1} dt = \lim_{x\to\infty} \int_1^{e^x} \frac{1}{1+u^2} du = \lim_{x\to\infty} [\tan^{-1} u]_1^{e^x} = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$.

   (e) $\lim_{x\to\infty} \int_1^x p^t dt = \lim_{x\to\infty} \left[\frac{p^x - p}{\ln p}\right] = \frac{-p}{\ln p}$.

4. **(*)** $\int_a^\infty f(t)dt = \ell \Leftrightarrow \ \forall\, \epsilon > 0 \ \exists\, N \geq a$ such that $\left|\int_a^x f(t)dt - \ell\right| < \epsilon$ for every $x \geq N$. (1)
   Suppose $\int_a^\infty f(t)dt = \ell$ then by (1), $\left|\int_x^y f(t)dt\right| < 2\epsilon$ for every $x, y \geq N$.
   Let us show the converse. Let $\epsilon > 0$. By the assumption,
   there exists $N \geq a$ such that $\left|\int_x^y f(t)dt\right| < \epsilon$ for every $x, y \geq N$. (2)
   Let $x_n \to \infty$. Use (2) to show that the sequence $\left(\int_a^{x_n} f(t)dt\right)$ satisfies the Cauchy criterion and let $\int_a^{x_n} f(t)dt \to L$ for some L. Hence there exists $x_{n_1} > N$ such that
   $|\int_a^{x_{n_1}} f(t)dt - L| < \frac{\epsilon}{2}$. Now for any $x > N$,
   $|\int_a^x f(t)dt - L| \leq |\int_a^x f(t)dt - \int_a^{x_{n_1}} f(t)dt| + |\int_a^{x_{n_1}} f(t)dt - L| < \epsilon$.

5. Let $\alpha = \sum_{n=1}^\infty (-1)^{n+1} \frac{1}{n}$. Observe that
   $\lim_{n\to\infty} \int_0^n f(t)dt = \lim_{n\to\infty} \left(\int_0^1 f(t)dt + \int_1^2 f(t)dt + ... + \int_{n-1}^n f(t)dt\right) = \alpha$
   and for $x \in [n, n+1]$, $\left|\alpha - \int_0^x f(t)dt\right| \leq \max\left\{\left|\alpha - \int_0^n f(t)dt\right|, \left|\alpha - \int_0^{n+1} f(t)dt\right|\right\}$.

6. Trivial

7. (a) **(*)** Note that, since f is decreasing, for $t \in (n, n+1)$, $f(n+1) \leq f(t) \leq f(n)$ and hence $f(n+1) \leq \int_n^{n+1} f(t)dt \leq f(n)$. Now
   $\mu(n+1) - \mu(n) = f(n+1) - \int_n^{n+1} f(t)dt \leq 0$ and
   $\mu(n) = \sum_{k=1}^n f(k) - \left(\sum_{k=1}^{n-1} \int_k^{k+1} f(t)dt\right) \geq \sum_{n=1}^n f(k) - \sum_{n=1}^{n-1} f(k) = f(n) > 0.$

(b) Follows from (a).

8. (a) Trivial.

   (b) Yes. The graph of such a function is given in Figure 1.

9. Note that $\frac{kt}{1+t^2} - \frac{1}{2t} = \frac{(2k-1)t^2-1}{2t(1+t^2)}$. When $k = \frac{1}{2}$, use the LCT with $\frac{1}{t^3}$ and when $k \neq \frac{1}{2}$ use the LCT with $\frac{1}{t}$.

10. **(*)** Let $\epsilon > 0$. Since f is decreasing and $f(t) \to 0$ as $t \to \infty$, there exists $N > 0$ such that $|f(t)| \leq \frac{\epsilon}{2M}$ for all $t \geq N$. Let $y > x > N$. Then by the second MVT for integrals, there exists $c \in [x, y]$ such that $\left|\int_x^y f(t)g(t)dt\right| = \left|f(c)\int_x^y g(t)dt\right| \leq |f(c)| \left|\int_a^y g(t)dt - \int_a^x g(t)dt\right| \leq \frac{\epsilon}{2M}2M = \epsilon$. By the Cauchy Criterion (Problem 4), $\int_a^\infty f(t)g(t)dt$ converges.

11. (a) For $p > 0$, $\int_1^\infty \frac{\sin t}{t^p}dt$ converges by the Dirichlet test. For $p \leq 0$, let $q = -p$. Then $\int_1^\infty t^q \sin t dt$ does not converge. If so, then its partial integral is bounded and hence again by the Dirichlet test $\int_1^\infty \frac{t^q \sin t}{t^q}dt$ converges.

    (b) Let $p > 1$ and $1 < q < p$. Then $\frac{(\ln t)/t^p}{1/t^q} = \frac{\ln t}{t^{p-q}} \to 0$ as $t \to \infty$. Therefore by the LCT, the integral converges. For $p \leq 1$, $\frac{(\ln t)/t^p}{1/t^p} = \ln t \to \infty$ as $t \to \infty$. Therefore by the LCT, the integral diverges for $p \leq 1$.

    (c) Consider $I_1 = \int_0^1 \frac{t^{p-1}}{1+t}dt$ and $I_2 = \int_1^\infty \frac{t^{p-1}}{1+t}dt$. For convergence of I_1, use the LCT with t^{p-1}. This shows that I_1 converges for $1 - p < 1$; that is $p > 0$. For the convergence of I_2, use the LCT with t^{p-2}. This shows that I_2 converges for $p < 1$. Therefore I converges only for $0 < p < 1$.

    (d) Let $p \in \mathbb{R}$. Use the LCT with $\frac{1}{t^2}$. Hence $\int_1^\infty t^p e^{-t}dt$ converges for all $p \in \mathbb{R}$.

    (e) Observe that $1 - \cos t$ behaves like $\frac{t^2}{2}$ near 0. So use the LCT with $\frac{1}{t^{p-2}}$ and observe that the integral converges for $p < 3$ and diverges for $p \geq 3$.

12. (a) If $\ell < 1$ then find $\epsilon > 0$ such that $\ell + \epsilon < 1$. Then there exists $N \in \mathbb{N}$ such that $|f(t)|^{\frac{1}{t}} \leq \ell + \epsilon$ for all $t \geq N$. That is $|f(t)| \leq (\ell + \epsilon)^t$ for all $t \geq N$. By Problem 3(e) and the comparison test, the integral converges absolutely.

    (b) If $\ell > 1$, then there exists $N \in \mathbb{N}$ such that $|f(t)|^{\frac{1}{t}} > 1$ for all $t \geq N$. That is $|f(t)| > 1$ for all $t \geq N$. This show that the integral diverges.

13. (a) Converges : Use the LCT with $\frac{1}{\sqrt{t}}$.

    (b) Converges : Write $\int_0^{\frac{\pi}{2}} \ln(\sin t)dt = \int_0^{\frac{\pi}{2}} [\ln(\frac{\sin t}{t}) + \ln t]dt$. Note that $\int_0^{\frac{\pi}{2}} \ln(\frac{\sin t}{t})dt$ is proper integral and use Problem 3(a).

    (c) Converges : Write $\int_0^\infty \frac{1}{t^2+\sqrt{t}}dt = \int_0^1 \frac{1}{t^2+\sqrt{t}}dt + \int_1^\infty \frac{1}{t^2+\sqrt{t}}dt$. Observe that $\frac{1}{t^2+\sqrt{t}} \leq \frac{1}{\sqrt{t}}$ and $\frac{1}{t^2+\sqrt{t}} \leq \frac{1}{t^2}$.

    (d) Converges : Use the LCT test with $\frac{1}{\sqrt{t}}$.

    (e) Converges : Note that $\int_0^1 \sin t^3 dt$ converges. For the convergence of $\int_1^\infty \sin t^3 dt$, take $u = t^3$ and use the Dirichlet test for $\int_1^\infty (3u^{\frac{3}{2}})^{-1} \sin u du$.

    (f) Converges : Observe that, for $x > a$, $\left|\int_a^x e^{\sin t} \sin 2t dt\right| \leq 8e$ and use the Dirichlet test.

    (g) Converges : Using the substitution $u = t^2$ leads to the integral $\frac{1}{2}\int_1^\infty \sin u^2 du$.

    (h) Diverges : Use the LCT with $\frac{1}{t^3}$.

(i) Converges absolutely : Use the comparison test with $\frac{6}{t^2}$.

(j) Diverges: Observe that $\sqrt{1-\cos t} = \sqrt{2}\sin\frac{t}{2}$ and use the LCT with $\frac{1}{t}$.

(k) Diverges: Apply the Root test.

(l) Converges: Apply the Root test.

14. Let $f(t) = e^{-t}t^{p-1}$. Suppose $I_1 = \int_0^1 f(t)dt$ and $I_2 = \int_1^\infty f(t)dt$. By Problem 11 (d), I_2 converges for all $p \in (0,\infty)$. If $p \geq 1$, then f is bounded on $(0,1]$ and hence I_1 converges. If $p < 1$, use LCT with $\frac{1}{t^{1-p}}$ and verify that I_1 converges for $1-p < 1$; that is for $p > 0$.