Practice problems 5: Continuity, Existence of points of maximum and minimum

1. Let $f : \mathbb{R} \to \mathbb{R}$ be such that for every $x, y \in \mathbb{R}$, $|f(x) - f(y)| \leq |x - y|$. Show that f is continuous at every point in $\mathbb{R}$.

2. Let $f(x) = |x|$ for every $x \in \mathbb{R}$. Show that f is continuous at every point in $\mathbb{R}$.

3. Let $f, g : \mathbb{R} \to \mathbb{R}$. Suppose g is continuous at x_0 and f is contiuous at $g(x_0)$ then $(f \circ g)$ is continuous at x_0.

4. If $f : \mathbb{R} \to \mathbb{R}$ is continuous at x_0, show that the function $|f|$, defined by $|f|(x) = |f(x)|$ for all $x \in \mathbb{R}$, is also continuous at x_0

5. Let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^{\frac{1}{k}}$ for some $k \in \mathbb{N}$. Show that f is continuous at every point in $(0, \infty)$.

6. Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = [x]$ where $[\cdot]$ is the function defined as in Problem 12 of PP1 and $g(x) = x - 2$ for all $x \in \mathbb{R}$. Show that (fg) is continuous at 2 whereas f is not continuous at 2 and g is continuous at 2.

7. Let $f : [0, \pi] \to \mathbb{R}$ be defined by $f(0) = 0$ and $f(x) = x \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$ for $x \neq 0$. Is f continuous at 0?

8. Let $f, g : \mathbb{R} \to \mathbb{R}$ be continuous such that given any two points $x_1 < x_2$, there exists a point x_3 such that $x_1 < x_3 < x_2$ and $f(x_3) = g(x_3)$. Show that $f(x) = g(x)$ for all $x \in \mathbb{R}$.

9. Let $f : (-1, 1) \to \mathbb{R}$ be a continuous function such that in every neighborhood of 0, there exists a point where f takes the value 0. Show that $f(0) = 0$.

10. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous at x_0 and $f(x_0) > 0$. Show that there is $\delta > 0$ such that $f(x) > 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$. Further, show that the function $\frac{1}{f}$ is defined on $(x_0 - \delta, x_0 + \delta)$ and continuous at x_0.

11. Let $f : \mathbb{R} \to \mathbb{R}$ satisfy $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. If f is continuous at 0, show that f is continuous at every point $c \in \mathbb{R}$.

12. Let $f : \mathbb{R} \to (0, \infty)$ satisfy $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Suppose f is continuous at 0. Show that f is continuous at all $x \in \mathbb{R}$.

13. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x) = f(x^2)$ for all $x \in \mathbb{R}$. Show that f is constant.

14. Let $f : [a, b] \to \mathbb{R}$ be continuous and $f(x) > 0$ for all $x \in [a, b]$. Show that there exists $\alpha > 0$ such that $f(x) \geq \alpha$ for all $x \in [a, b]$.

15. Let $f : [0, 1] \to (0, 1)$ be an on-to function. Show that f is not continuous on $[0, 1]$.

16. **(*)** Let $f : [a, b] \to \mathbb{R}$ be one-one and continuous.

    (a) Show that f is not on-to.

    (b) Show that f^{-1} is continuous on $\{f(x) : x \in [a, b]\}$, the range of f.

17. **(*)** Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. Show that $f(x) = f(1)x$ for all $x \in \mathbb{R}$.

Please write to psraj@iitk.ac.in if any typos/mistakes are found in this set of practice problems/solutions/hints.

18. **(*)** Let $f : (0,1) \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ where } p, q \in \mathbb{N} \text{ and } \; p, q \text{ have no common factors} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

(a) Let $x_n = \frac{p_n}{q_n} \in (0,1)$ where $p_n, q_n \in \mathbb{N}$ and have no common factors. Suppose $x_n \to x$ for some x with $x_n \neq x$ for all $n \in \mathbb{N}$. Show that $q_n \to \infty$.

(b) Show that f is continuous at every irrational.

(c) Show that f is discontinuous at every rational.

Practice Problems 5: Hints/solutions

1. Let $x_0 \in \mathbb{R}$ and $x_n \to x_0$. Since $\mid f(x_n) - f(x_0) \mid \leq \mid x_n - x_0 \mid, f(x_n) \to f(x_0)$. Hence f is continuous at x_0. Since x_0 is arbitrary, f is continuous at every point in $\mathbb{R}$.

2. Use Problem 1.

3. Let $x_n \to x_0$. Since g is continuous at x_0, $g(x_n) \to g(x_0)$. Since f is continuous at $g(x_0)$, $f(g(x_n)) \to f(g(x_0))$, i.e., $(f \circ g)(x_n) \to (f \circ g)(x_0)$. Hence $(f \circ g)$ is continuous at x_0.

4. Observe that $|f|(x) = |f(x)| = (|\cdot| \circ f)(x)$. Apply Problems 2 and 3.

5. Use Problem 7 of PP2.

6. Let $x_n \to 2$. Since (x_n) is bounded, $([x_n])$ is also bounded. Therefore, there exists $M > 0$ such that $|[x_n]| \leq M$ for all $n \in \mathbb{N}$. Since $|(fg)(x_n)| = |f(x_n)g(x_n)|$, we have $|(fg)(x_n)| \leq M|x_n - 2|$. This shows that $(fg)(x_n) \to 0 = (fg)(2)$.

7. The function f is not continuous at 0, because, $x_n = \frac{1}{2n\pi} \to 0$ but $f(\frac{1}{2n\pi}) \nrightarrow f(0)$.

8. Fix some $x_0 \in \mathbb{R}$. For every n, find x_n such that $x_0 - \frac{1}{n} < x_n < x_0$ and $(f - g)(x_n) = 0$. Since $x_n \to x_0$, by the continuity of $f - g$, $(f - g)(x_n) \to (f - g)(x_0) = 0$.

9. For every n, find $x_n \in (-\frac{1}{n}, \frac{1}{n})$ such that $f(x_n) = 0$. Since f is continuous at 0 and $x_n \to 0$, we have $f(x_n) \to f(0)$. Therefore, $f(0) = 0$.

10. Let $f(x_0) > 0$. Choose $\epsilon = \frac{f(x_0)}{2}$. Then there exists $\delta > 0$, such that $f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$ whenever, $x \in (x_0 - \delta, x_0 + \delta)$. Hence $f(x) > \frac{f(x_0)}{2}$ for all $x \in (x_0 - \delta, x_0 + \delta)$. The continuity of $\frac{1}{f}$ at x_0 follows from Theorem 2.1 and the continuity of f at x_0.

11. First note that $f(0) = 0, f(-x) = -f(x)$ and $f(x - y) = f(x) - f(y)$. Let $x_0 \in \mathbb{R}$ and $x_n \to x_0$. Since f is continuous at 0 and $x_n - x_0 \to 0$ we have $f(x_n) - f(x_0) = f(x_n - x_0) \to f(0) = 0$.

12. Since $f(0) = f(0)^2, f(0) = 1$ and since $f(x - x) = f(0), f(-x) = \frac{1}{f(x)}$. Let $x_0 \in \mathbb{R}$ and $x_n \to x_0$. By the continuity of f at 0, $f(x_n - x_0) \to 1$ and hence $f(x_n) \to \frac{1}{f(-x_0)} = f(x_0)$.

13. Suppose $x > 0$. By the assumption, $f(x) = f(x^{\frac{1}{2}}) = f(x^{\frac{1}{2^2}}) = f(x^{\frac{1}{2^n}})$. Since $x^{\frac{1}{2^n}} \to 1, f(x^{\frac{1}{2^n}}) \to f(1)$, i.e. $f(x) = f(1)$. Now $f(-x) = f((-x)^2) = f(x^2) = f(x)$. At $x = 0$, by the continuity of f at 0, $f(\frac{1}{n}) \to f(0)$. Since $f(\frac{1}{n}) = f(1)$, $f(0) = f(1)$. Therefore $f(x) = f(1)$ for all $x \in \mathbb{R}$.

14. By Theorem 5.2, f bounded. Hence, let $\alpha = inf\{f(x) : x \in [a, b]\}$. By Theorem 5.3, there exists $x_0 \in [a, b]$ such that $f(x_0) = \alpha$. Since $f(x_0) > 0$, $\alpha > 0$.

15. Note that $\inf\{f(x) : x \in [a, b]\} = 0$. If f is continuous, then by Theorem 5.3, there exists $x_0 \in [a, b]$ such that $f(x_0) = 0$.

16. (a) By Theorem 5.2, f is bounded on $[a, b]$. Hence f is not on-to.

    (b) Let $f(x_n) \to f(x_0)$ for some $x_n, x_0 \in [a, b]$. We show that $x_n \to x_0$ which proves that f^{-1} is continuous at $f(x_0)$. If (x_{n_k}) is any subsequence of (x_n), then by the Bolzano-Weierstrass theorem, there exists a subsequence $(x_{n_{k_i}})$ such that $x_{n_{k_i}} \to \alpha$ for some $\alpha \in [a, b]$. By the continuity of f, $f(x_{n_{k_i}}) \to f(\alpha)$. By our assumption $f(\alpha) = f(x_0)$. Since f is one-one, $x_0 = \alpha$. By Problem 11 of PP3, $x_n \to x_0$.

17. First observe that $f(0) = 0$ and $f(n) = nf(1)$ for all $n \in \mathbb{N}$. Next note that $f(-1) = -f(1)$ and $f(m) = f(1)m$ for all $m \in \mathbb{Z}$. By observing $f(\frac{1}{n}) = f(1)\frac{1}{n}$ for all $n \in \mathbb{N}$, show that $f(\frac{m}{n}) = f(1)\frac{m}{n}$ for all $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Finally take any irrational number x and find $r_n \in \mathbb{Q}$ such that $r_n \to x$. Apply the continuity to conclude that $f(x) = f(1)x$.

18. (a) If for some $M \in \mathbb{N}$, $q_n < M$ for all $n \in \mathbb{N}$, then the set $\{x_n : n \in \mathbb{N}\}$ is finite which is not true. Similarly we can show that any subsequence of (q_n) cannot be bounded.

    (b) Suppose x_0 is irrational in $(0, 1)$ and $x_n \to x_0$ where x_n can be rational or irrational. Apply (a) to show that $f(x_n) \to 0 = f(x_0)$.

    (c) Suppose x_0 is rational in $(0, 1)$. To show that f is discontinuous at x_0, choose an irrational sequence (x_n) such that $x_n \to x_0$.