Practice Problems 9: Taylor's Theorem

1. Let $f : [a,b] \to \mathbb{R}$ and $n \in \mathbb{N}$. Suppose that $f^{(n+1)}$ exists on $[a,b]$ and $f^{(n+1)}(x) = 0$ for all $x \in [a,b]$. Show that f is a polynomial of degree less than or equal to n.

2. Show that $1 + \frac{x}{2} - \frac{x^2}{8} \le \sqrt{1+x} \le 1 + \frac{x}{2}$ for $x > 0$.

3. Show that for $x > 0$, $|\log(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3}\right)| \le \frac{x^4}{4}$.

4. Using Taylor's theorem, show that $1 - \frac{1}{2}x^2 \le \cos x$ for all $x \in \mathbb{R}$.

5. Let $x \in \mathbb{R}$ be such that $|x|^5 < \frac{5!}{10^4}$. Show that $\sin x$ can be approximated by $x - \frac{x^3}{6}$ with an error of magnitude less than or equal to 10^{-4}.

6. Using Taylor's theorem, establish the binomial expansion:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + ... + x^n, \; x \in \mathbb{R}.$$

7. Using Taylor's theorem, compute $\lim\limits_{x \to 0} \frac{1-\sqrt{1+x^2}\cos x}{x^4}$.

8. Using the EMVT show that $\cos y - \cos x \ge (x-y)\sin x$ for all $x, y \in [\frac{\pi}{2}, \frac{3\pi}{2}]$.

9. (a) Let $f : [a,b] \to \mathbb{R}$ be such that $f''(x) \ge 0$ for all $x \in [a,b]$. Suppose that $x, y \in (a,b)$, $x < y$ and $0 < \lambda < 1$. Show that

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$$

i.e., the chord joining the two points $(x, f(x))$ and $(y, f(y))$ lies above the portion $\{(t, f(t)) : t \in (x,y)\}$ of the graph.

(b) Show that $\lambda \sin x \le \sin \lambda x$ for all $x \in [0, \pi]$ and $0 < \lambda < 1$.

10. Let $f : [a,b] \to \mathbb{R}$ be twice differentiable. Suppose $f'(a) = f'(b) = 0$. Show that there exist $c_1, c_2 \in (a,b)$ such that $|f(b) - f(a)| = \left(\frac{b-a}{2}\right)^2 \frac{1}{2} \, |f''(c_1) - f''(c_2)|$.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f'''(x) > 0$ for all $x \in \mathbb{R}$. Suppose that $x_1, x_2 \in \mathbb{R}$ and $x_1 < x_2$. Show that $f(x_2) - f(x_1) > f'\left(\frac{x_1+x_2}{2}\right)(x_2 - x_1)$.

12. Let f be a twice differentiable function on $\mathbb{R}$ such that $f''(x) \ge 0$ for all $x \in \mathbb{R}$. Show that if f is bounded then it is a constant function.

13. (a) For a positive integer n, show that there exists $c \in (0,1)$ such that

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{e^c}{(n+1)!}.$$

Further, show that $\frac{e^c}{n+1} = n!e - m$ for some integer m.

(b) **(*)** Show that e is an irrational number.

Please write to psraj@iitk.ac.in if any typos/mistakes are found in this set of practice problems/solutions/hints.

14. **(*) (Taylor's theorem with the Cauchy remainder)** Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n)}$ be continuous on $[a, b]$ and $f^{(n+1)}$ exists on (a, b). Suppose $x_0 \in [a, b]$ and $x \in [a, b] \backslash \{x_0\}$. For every $t \in [a, b]$, define

$$g(t) = f(x) - f(t) - (x - t)f'(t) - \frac{(x - t)^2}{2!} f''(t) - \cdots - \frac{(x - t)^n}{n!} f^{(n)}(t).$$

(a) Show that $g'(t) = -\frac{(x-t)^n}{n!} f^{(n+1)}(t)$.

(b) Show that there exists c between x and x_0 such that $\frac{g(x)-g(x_0)}{x-x_0} = -\frac{(x-c)^n}{n!} f^{(n+1)}(c)$.

(c) Show that there exists c between x and x_0 such that

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!} f''(x_0) + \cdots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + \frac{(x - c)^n (x - x_0)}{n!} f^{(n+1)}(c).$$

Practice Problems 9: Hints/Solutions

1. Fix $x \in (a, b]$. By Taylor's Theorem, $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{(n)}(a)}{n!}(x-a)^n$ which is a polynomial of degree $\leq n$.

2. By Taylor's theorem, there exists $c \in (0, x)$ such that $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{1}{8}\frac{x^2}{(1+c)^{3/2}}$.

3. By Taylor's theorem, there exists $c \in (0, x)$ such that $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4(1+c)^4}$.

4. By Taylor's Theorem, there exists c between 0 and x such that $\cos x = 1 - \frac{1}{2}x^2 + \frac{\sin c}{6}x^3$. Verify that $\frac{\sin c}{6}x^3 \geq 0$ when $| x | \leq \pi$. If $| x | \geq \pi$ then $1 - \frac{1}{2}x^2 < -3 \leq \cos x$.

5. By Taylor's theorem, there exists c between 0 and x such that $\sin x = x - \frac{x^3}{3!} + (\cos c)\frac{x^5}{5!}$. If $|x|^5 < \frac{5!}{10^4}$, then $| \sin x - \left(x - \frac{x^3}{6}\right) | \leq 10^{-4}$.

6. Let $f(x) = x^n$. By Taylor's theorem there exists c between 1 and $1+x$ such that $(1+x)^n = f(1) + f'(1)x + \frac{f''(1)}{2!}x^2 + \cdots + \frac{f^n(1)}{n!}x^n + \frac{f^{n+1}(c)}{(n+1)!}x^{n+1}$ which leads to the answer.

7. Observe from Taylor's theorem that $\sqrt{1+x^2} = 1 + \frac{x^2}{2} - \frac{x^4}{8} + \alpha x^6$ and $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \beta x^5$ for some α and β in $\mathbb{R}$. The limit is $\frac{1}{3}$.

8. Let $x, y \in [\frac{\pi}{2}, \frac{3\pi}{2}]$. By the EMVT, there exists c between x and y such that $\cos y = \cos x - (y-x)\sin x - \frac{\cos c}{2}(y-x)^2$. This leads to the answer.

9. (a) Let $x_\lambda = \lambda x + (1-\lambda)y$. Since $f''(t) \geq 0$ for all $t \in [a, b]$, by the EMVT, $f(x) \geq f(x_\lambda) + f'(x_\lambda)(1-\lambda)(x-y)$ and $f(y) \geq f(x_\lambda) + f'(x_\lambda)\lambda(y-x)$. Eliminate $f'(x_\lambda)$.

   (b) Define $f(x) = -\sin x$ on $[0, \pi]$. Take $y = 0$ and apply the inequality given in (a).

10. By the EMVT theorem, $f\left(\frac{a+b}{2}\right) = f(a) + \frac{f''(c_1)}{2}\left(\frac{b-a}{2}\right)^2$ and $f\left(\frac{a+b}{2}\right) = f(b) + \frac{f''(c_2)}{2}\left(\frac{b-a}{2}\right)^2$ for some $c_1, c_2 \in (a, b)$. Eliminate $f\left(\frac{a+b}{2}\right)$.

11. Let $\overline{x} = \frac{x_1+x_2}{2}$. Since $f'''(x) > 0$ for all $x \in \mathbb{R}$, by Taylor's theorem $f(x_2) > f(\overline{x}) + f'(\overline{x})(x_2 - \overline{x}) + \frac{f''(\overline{x})}{2}(x_2 - \overline{x})^2$ and $f(x_1) < f(\overline{x}) + f'(\overline{x})(x_1 - \overline{x}) + \frac{f''(\overline{x})}{2}(x_2 - \overline{x})^2$. Eliminate $f(\overline{x})$ and $\frac{f''(\overline{x})}{2}(x_2 - \overline{x})^2$.

12. Suppose $f'(x_0) > 0$ for some $x_0 \in \mathbb{R}$. Since $f''(x) \geq 0$ for all $x \in \mathbb{R}$, by the EMVT, $f(x) \geq f(x_0) + f'(x_0)(x - x_0) \to \infty$ as $x \to \infty$. If $f'(x_0) < 0$, then $f(x) \to \infty$ as $x \to -\infty$. This contradicts the fact that f is bounded.

13. (a) Let $f(x) = e^x$ on $[0, 1]$. By Taylor's theorem, there exists $c \in (0, 1)$ such that $e = 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{e^c}{(n+1)!}$. Multiply both sides by $n!$ to get $\frac{e^c}{n+1} = n!e - m$ for some integer m.

    (b) If $e = \frac{p}{q}$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, then by (a), $\left(\frac{p}{q}\right)^c \frac{1}{n+1} = n!\frac{p}{q} - m$. Since $n!\frac{p}{q} - m$ is an integer for $n \geq q$, $\left(\frac{p}{q}\right)^c \frac{1}{n+1}$ is a natural number for every $n \geq q$. But $\left(\frac{p}{q}\right)^c \frac{1}{n+1} \to 0$ as $n \to \infty$ which is a contradiction.

14. (a) Observe that for $k \geq 1$, $\frac{d}{dt}\left(-\frac{(x-t)^k}{k!}f^{(k)}(t)\right) = \frac{(x-t)^{k-1}}{(k-1)!}f^{(k)}(t) - \frac{(x-t)^k}{k!}f^{(k+1)}(t)$.

    (b) Use (a) and the MVT.

    (c) Since $g(x) = 0$, by (b), $g(x_0) = \frac{(x-c)^n(x-x_0)}{n!}f^{(n+1)}(c)$ which establishes part (c).