MTH 112M - 2024

Assignment 1: Polar co-ordinates, Applications of Integration,

1. Find the area of the region in the first quadrant bounded on the left by the Y-axis, below by the curve $x = 2\sqrt{y}$, above left by the curve $x = (y-1)^2$, and above right by the line $x = 3-y$.

2. Sketch the graphs of $r = -|\cos\theta|$ and $r^2 = -\cos\theta$.

3. Sketch the graphs of $r = \cos(2\theta)$ and $r = \sin(2\theta)$. Also, find their points of intersection.

4. Sketch the graph of $r = 1+\sin\theta$. Find the area of the region that is inside the circle $r = 3\sin\theta$ and also inside $r = 1+\sin\theta$.

5. A curved wedge is cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a $45°$ angle at the center of the cylinder. Find the volume of the wedge.

6. Let C denote the circular disc of radius b centered at $(a, 0)$ where $0 < b < a$. Find the volume of the torus that is generated by revolving C around the y-axis using
   (a) the Washer Method
   (b) the Shell Method.

7. Consider the curve C defined by $x(t) = \cos^3(t),\ y(t) = \sin^3 t,\ 0 \le t \le \frac{\pi}{2}$.
   (a) Find the length of the curve.
   (b) Find the area of the surface generated by revolving C about the x-axis.

Assignment 2: Vectors, Curves, Surfaces, Vector Functions

1. Consider the planes $x - y + z = 1,\ x + ay - 2z + 10 = 0$ and $2x - 3y + z + b = 0$, where a and b are parameters. Determine the values of a and b such that the three planes
   (a) intersect at a single point,
   (b) intersect in a line,
   (c) intersect (taken two at a time) in three distinct parallel lines.

2. Sketch the surfaces by sketching the cross sections cut from the surfaces by the planes $x = 0, y = 0$ and $z = 1$.
   (a) $z = x^2$ (Cylinder)
   (b) $x^2 + y^2 = 4$ (Circular Cylinder)
   (c) $4z = x^2 + y^2$ (Paraboloid)
   (d) $4z^2 = x^2 + y^2$ (Circular cone(s))

3. Sketch the following parametric curves:
   (a) $R_1(t) = (\cos t, \sin t, t), t \in \mathbb{R}$
   (b) $R_2(t) = (t\cos t, t\sin t, t), t \in \mathbb{R}$
   (c) $R_3(t) = (t\cos t, t\sin t, t^2), t \ge 0$
   (d) $R_4(t) = (\cos^2 t, \sin^2 t, t), t \ge 0$
   (e) $R_5(t) = (t\cos t, t\sin t), t \ge 0$

4. The velocity of a particle moving in space is $\frac{d}{dt}c(t) = (\cos t)\vec{i} - (\sin t)\vec{j} + \vec{k}$. Find the particle's position as a function of t if $c(0) = 2\vec{i} + \vec{k}$. Also find the angle between its position vector and the velocity vector.

5. Show that $c(t) = \sin t^2\vec{i} + \cos t^2\vec{j} + 5\vec{k}$ has constant magnitude and is orthogonal to its derivative. Is the velocity vector of constant magnitude?

6. Find the point on the curve $c(t) = (5\sin t)\vec{i} + (5\cos t)\vec{j} + 12t\vec{k}$ at a distance 26π units along the curve from $(0, 5, 0)$ in the direction of increasing arc length.

7. Reparametrize the curves

   (a) $c(t) = \frac{t^2}{2}\vec{i} + \frac{t^3}{3}\vec{k}, \quad 0 \le t \le 2,$

   (b) $c(t) = 2\cos t\vec{i} + 2\sin t\vec{j}, \quad 0 \le t \le 2\pi$

   in terms of arc length.

Assignment 3: Functions of several variables (Continuity and Differentiability)

1. Sketch the graphs of the following functions:

   (a) $F_1(x, y) = x^2 + y^2$, $(x, y) \in \mathbb{R}^2$.

   (b) $F_2(x, y) = \sqrt{x^2 + y^2}$, $(x, y) \in \mathbb{R}^2$.

2. Sketch the (level) surfaces defined as follows:

   (a) $F_1(x, y, z) = 2$ where $F_1(x, y, z) = z - (x^2 + y^2)$ for all $(x, y, z) \in \mathbb{R}^3$.

   (b) $F_2(x, y, z) = 0$ where $F_2(x, y, z) = z^2 - (x^2 + y^2)$ for all $(x, y, z) \in \mathbb{R}^3$.

   (c) $F_3(x, y, z) = 4$ where $F_3(x, y, z) = x^2 + y^2$ for all $(x, y, z) \in \mathbb{R}^3$.

3. Identify the points, if any, where the following functions fail to be continuous:

$$(i)\ \ f(x,y) = \begin{cases} xy & \text{if } xy \ge 0 \\ -xy & \text{if } xy < 0 \end{cases} \qquad (ii)\ \ f(x,y) = \begin{cases} xy & \text{if } xy \text{ is rationnal} \\ -xy & \text{if } xy \text{ is irrational.} \end{cases}$$

4. Consider the function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$f(x,y) = \begin{cases} \frac{x^2y^2}{x^2y^2+(x-y)^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if} (x,y) = (0,0) \end{cases}$$

   Show that the function satisfy the following:

   (a) The iterated limits $\lim_{x\longrightarrow 0}\left[\lim_{y\longrightarrow 0} f(x,y)\right]$ and $\lim_{y\longrightarrow 0}\left[\lim_{x\longrightarrow 0} f(x,y)\right]$ exist and equals 0;

   (b) $\lim_{(x,y)\longrightarrow(0,0)} f(x,y)$ does not exist;

   (c) $f(x, y)$ is not continuous at $(0, 0)$;

   (d) the partial derivatives exist at $(0, 0)$.

5. Let $f(x, y) = (x^2+y^2)\sin\frac{1}{x^2+y^2}$ if $(x, y) \neq (0, 0)$ and 0, otherwise. Show that f is differentiable at every point of $\mathbb{R}^2$ but the partial derivatives are not continuous at $(0, 0)$.

6. Let $f(x, y) = |xy|$ for all $(x, y) \in \mathbb{R}^2$. Show that

(a) f is differentiable at $(0,0.)$

(b) $f_x(0,y_0)$ does not exist if $y_0 \neq 0$.

Assignment 4: Directional derivatives, Maxima, Minima, Lagrange Multipliers

1. Let $f(x,y) = \frac{1}{2}\Big(\big|\,|x|-|y|\,\big|-|x|-|y|\Big)$. Is f continuous at $(0,0)$? Which directional derivatives of f exist at $(0,0)$? Is f differentiable at $(0,0)$?

2. Let $f(x,y) = \frac{x^2y}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0) = 0$. Show that the directional derivative of f at $(0,0)$ in all directions exist but f is not differentiable at $(0,0)$.

3. Let $f(x,y) = x^2e^y + \cos(xy)$. Find the directional derivative of f at $(1,2)$ in the direction $(\frac{3}{5}, \frac{4}{5})$.

4. Find the equation of the surface generated by the normals to the surface $x + 2yz + xyz^2 = 0$ at all points on the z-axis.

5. Examine the following functions for local maxima, local minima and saddle points:

   i) $4xy - x^4 - y^4$ *ii)* $x^3 - 3xy^2$

Assignment 5 : Double Integrals

1. Evaluate the following integrals:

   i) $\int\limits_0^1 \int\limits_0^{\sqrt{1-x^2}} \sqrt{1-y^2}\,dydx$ *ii)* $\int\limits_0^{\pi}\int\limits_x^{\pi} \frac{\sin y}{y}\,dydx$ *iii)* $\int\limits_0^{1}\int\limits_y^{1} x^2 \exp^{xy}\,dxdy$.

2. Evaluate $\iint\limits_R x\,dxdy$ where R is the region $1 \leq x(1-y) \leq 2$ and $1 \leq xy \leq 2$.

3. Using double integral, find the area enclosed by the curve $r = sin3\theta$ given in polar cordinates.

4. Compute $\lim\limits_{a \longrightarrow \infty} \int\int\limits_{D(a)} \exp^{-(x^2+y^2)}\,dxdy$, where

   i) $D(a) = \{(x,y) : x^2+y^2 \leq a^2\}$ and *ii)* $D(a) = \{(x,y) : 0 \leq x \leq a,\ 0 \leq y \leq a\}$.

   Hence prove that (i) $\int\limits_0^{\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$ (ii) $\int\limits_0^{\infty} x^2e^{-x^2}dx = \frac{\sqrt{\pi}}{4}$.

5. Find the volume of the solid which is common to the cylinder $x^2+y^2 = 1$ and $x^2+z^2 = 1$.

Assignment 6 : Triple Integrals, Surface Integrals, Line integrals

1. Evaluate the integral $\iiint\limits_W \frac{dzdydx}{\sqrt{1+x^2+y^2+z^2}}$; where W is the ball $x^2+y^2+z^2 \leq 1$.

2. What is the integral of the function x^2z taken over the entire surface of a right circular cylinder of height h which stands on the circle $x^2+y^2 = a^2$. What is the integral of the given function taken throughout the volume of the cylinder.

3. Find the line integral of the vector field $F(x,y,z) = y\vec{i} - x\vec{j} + \vec{k}$ along the path $\mathbf{c}(t) = (\cos t, \sin t, \frac{t}{2\pi})$, $0 \leq t \leq 2\pi$ joining $(1,0,0)$ to $(1,0,1)$.

4. Evaluate $\int_C T \cdot dR$, where C is the circle $x^2 + y^2 = 1$ and T is the unit tangent vector.

5. Show that the integral $\int_C yzdx + (xz+1)dy + xydz$ is independent of the path C joining $(1, 0, 0)$ and $(2, 1, 4)$.

Assignment 7 : Green's /Stokes' /Gauss' Theorems

1. Use Green's Theorem to compute $\int_C (2x^2 - y^2)\, dx + (x^2 + y^2)\, dy$ where C is the boundary of the region $\{(x, y) : x, y \geq 0 \ \& \ x^2 + y^2 \leq 1\}$.

2. Use Stokes' Theorem to evaluate the line integral $\int_C -y^3\, dx + x^3\, dy - z^3\, dz$, where C is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + y + z = 1$ and the orientation of C corresponds to counterclockwise motion in the xy-plane.

3. Let $\overrightarrow{F} = \frac{\overrightarrow{r}}{|\overrightarrow{r}|^3}$ where $\overrightarrow{r} = x\,\overrightarrow{i} + y\,\overrightarrow{j} + z\,\overrightarrow{k}$ and let S be any surface that surrounds the origin. Prove that $\iint_S \overrightarrow{F}.n\; d\sigma = 4\pi$.

4. Let D be the domain inside the cylinder $x^2+y^2 = 1$ cut off by the planes $z = 0$ and $z = x+2$. If $\vec{F} = (x^2 + ye^z, y^2 + ze^x, z + xe^y)$, use the divergence theorem to evaluate $\iint_{\partial D} F \cdot \mathbf{n}\; d\sigma$.