Lecture 24 : Calculus of vector valued functions

In the previous lectures we had been dealing with functions from a subset of $\mathbb{R}$ to $\mathbb{R}$. In this lecture we will deal with the functions whose domain is a subset of $\mathbb{R}$ and whose range is in $\mathbb{R}^3$ (or $\mathbb{R}^n$). Such functions are called vector valued functions of a real variable.

If the values of a function F are in $\mathbb{R}^3$, then each $F(t)$ has 3 components, for example $F(t) = ((f_1(t), f_2(t), f_3(t))$. Therefore, each vector valued function F is associated with 3 real valued functions f_1, f_2 and f_3 and in this case we write $F = (f_1, f_2, f_3)$.

Let us see some examples of vector valued functions.

Examples: 1. Let $X_0, P \in \mathbb{R}^3$ and $P \neq 0$. Consider the vector valued function $F(t) = X_0 + tP$. It is clear that the range of the vector valued function is the line though the point X_0 parallel to the vector P.

2. Consider the vector valued functions $F_1(t) = (cost, sint),\ 0 \leq t \leq 2\pi$ and $F_2(t) = (cost, sint, t)$, $\infty < t < \infty$. We can geometrically visualize the ranges of F_1 and F_2 as t varies. In fact $F_1(t)$ varies on a circle and $F_2(t)$ varies on a helix. Both these curves are particular cases of parametric curves.

Parametric curves: Let I be an interval and $F : I \to \mathbb{R}^3$. The set of points $\{F(t) : t \in I\}$ is called the graph of the function F. If F is continuous (for the definition see below) then such a graph is called a curve or parametric curve with the parameter t.

From the previous definition it is clear that each continuous vector valued function corresponds to a curve. Naturally one expects that some geometric properties of the curves can be investigated by using some properties of the vector valued functions.

In this lecture we will extend the basic concepts of calculus, such as limit, continuity and derivative, to vector valued functions and see some applications to the study of curves.

Limits and derivatives:

Let $F = (f_1, f_2, f_3)$ be a vector valued function and $L = (l_1, l_2, l_3)$.

We say that $\lim_{t\to t_0} F(t) = L$ if $\lim_{t\to t_0} \parallel F(t) - L \parallel = 0$.

Proposition: $\lim_{t\to t_0} F(t) = L$ if and only if $\lim_{t\to t_0} f_i(t) = l_i$ for $i = 1, 2, 3$.

Proof: This follows from the fact that $\sum_{i=1}^{3} \mid f_i(t) - l_i \mid^2 \to 0 \ \Leftrightarrow\ \mid f_i(t) - l_i \mid \to 0,\ \ i = 1, 2, 3.$ $\square$

From the previous result it follows that $\lim_{t\to t_0} F(t) = (\lim_{t\to t_0} f_1(t), \lim_{t\to t_0} f_2(t), \lim_{t\to t_0} f_3(t))$ whenever the component on the right is meaningful.

We say that F is continuous at t_0 if $\lim_{t\to t_0} F(t) = F(t_0)$. One can show that F is continuous at t_0 if and only if each of the component function f_i is continuous at t_0.

We say that F is differentiable at t_0 if $\lim_{h\to 0} \frac{F(t_0+h)-F(t_0)}{h}$ exists. The limit is called the derivative of F at t_0 and is denoted by $F'(t_0)$. Note that F is differentiable at t_0 if and only if f_i is differentiable at t_0 for all $i = 1, 2, 3$. Moreover, $F'(t_0) = (f_1'(t_0), f_2'(t_0), f_3'(t_0))$.

Tangent Vector: As in the case of a real valued function, we will see that the derivative $F'(t_0)$ is related to the concept of tangency. Suppose F is differentiable at t_0 and $F'(t_0) \neq 0$. Then

$$F'(t_0) = \lim_{h\to 0} \frac{1}{h}(F(t_0 + h) - F(t_0)).$$

Geometrically, one can visualize that the vector $\frac{1}{h}(F(t_0+h)-F(t_0))$, which is parallel to the vector $F(t_0+h)-F(t_0)$, moves to be a tangent vector as $h \to 0$. In view of this we have the following definition.

Definition: Suppose C is a curve defined by a differentiable vector valued function R. Suppose $R'(t_0) \neq 0$. The vector $R'(t_0)$ is called a tangent vector to C at $F(t_0)$ and the line $X(t) = R(t_0) + tR'(t_0)$ is called the tangent line to C at $R(t_0)$.

Example: Let us find the equation of the plane perpendicular to the circular helix $R(t) = (\cos t, \sin t, t)$ at $t_0 = \frac{\pi}{3}$. The equation of the plane passing through $R(\frac{\pi}{3})$ and perpendicular to $R'(\frac{\pi}{3})$ is the required plane. So the plane is $R'(\frac{\pi}{3}) \cdot (x,y,z) = R'(\frac{\pi}{3}) \cdot R(\frac{\pi}{3})$.

Arc length for space curves: We have seen a formula for evaluating the length of a plane curve. The formula can be extended to the space curves. Let C be a space curve defined by $R(t) = x(t)i + y(t)j + z(t)k$, $a \leq t \leq b$. Throughout this lecture we will assume that R' is continuous. The length of the curve C is defined to be

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}dt = \int_a^b \parallel \frac{dR}{dt} \parallel .$$

Arc length parameter: Let $R(t_0)$ be a fixed point on the curve C. For t, the directed distance measured along C from $R(t_0)$ and up to $R(t)$ is $s(t) = \int_{t_0}^{t} \sqrt{x'(\tau)^2 + y'(\tau)^2 + z'(\tau)^2}d\tau$. Each value of s corresponds to a point on C and this parametrizes C with respect to s, the arc length parameter. By the first FTC we have,

$$\frac{ds}{dt} = \parallel \frac{dR}{dt} \parallel .$$

This is expected. Because, if we consider $R(t)$ is the position vector of a particle moving along C, then $v(t) = R'(t)$ is the velocity vector and $a(t) = v'(t)$ is the acceleration vector. The speed with which the particle moves along its path is the magnitude of v.

Unit tangent vector: The unit tangent vector of $R(t)$ is $T = \frac{R'(t)}{\|R'(t)\|}$ whenever $\parallel R'(t) \parallel \neq 0$. From the derivation of $\frac{ds}{dt}$, we get $T = \frac{\frac{dR}{dt}}{\frac{ds}{dt}}$. Now can we write

$$T = \frac{dR}{dt}\frac{dt}{ds} = \frac{dR}{ds} \quad ?$$

The second equality of the above equation follows from the chain rule and the first equation follows from the following theorem.

Theorem: Let $f : [a,b] \to \mathbb{R}$, $f'(x) \neq 0$ for all $x \in [a,b]$. Then f^{-1} is continuous, differentiable and $(f^{-1})'(f(x)) = \frac{1}{f'(x)}$.

Proof (*): Use the sequential argument and Bolzano-Weierstrass theorem to prove that f^{-1} is continuous. Let $f : [a,b] \to [c,d]$, $c \leq y_0 \leq d, y_0 = f(x_0)$ and $y = f(x)$ for some $x \in [a,b]$. Suppose $y \neq y_0$. Then,

$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{\frac{f(x)-f(x_0)}{x-x_0}}$$

Now let $y \to y_0$ and use the continuity of f^{-1} to get the result. □

Let us go back to the question we asked above. Let us work with the curve C such that $s(t)$ increases as $t > t_0$ increases, that is $\frac{ds}{dt} > 0$. By the previous theorem we have, $\frac{dt}{ds} = \frac{1}{\frac{ds}{dt}}$.