Lecture 25 : Principal Normal and Curvature

In the previous lecture we defined unit tangent vectors to space curves. In this lecture we will define normal vectors.

Consider the following results.

Theorem: If F and G are differentiable vector valued functions then so is $F \cdot G$ and $(F \cdot G)' = F' \cdot G + F \cdot G'$.

Proof : Let $F = (f_1, f_2, f_3)$ and $G = (g_1, g_2, g_3)$. Then $(F \cdot G)' = (f_1 g_1 + f_2 g_2 + f_3 g_3)'$. By simplifying this we get the result. $\square$

Theorem: Let I be an interval and F be a vector valued function on I such that $\| F(t) \| = \alpha$ for all $t \in I$. Then $F \cdot F' = 0$ on I, that is $F'(t)$ is perpendicular to $F(t)$ for each $t \in I$.

Proof: Let $g(t) = \| F(t) \|^2 = F(t) \cdot F(t)$. By assumption g is constant on I and therefore $g' = 0$ on I. By the previous theorem, $g' = 2F \cdot F'$. Therefore, $F \cdot F' = 0$. $\square$

Since the unit tangent vector T has constant length 1, by the previous theorem, T' is perpendicular to T. In view of this, we define the principle normal to the curve

$$N(t) = \frac{T'(t)}{\| T'(t) \|}$$

whenever, $\| T'(t) \| \neq 0$.

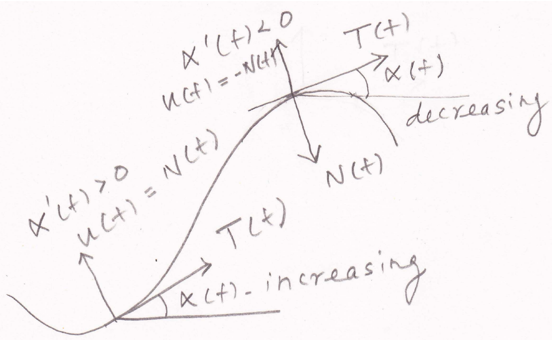
Geometric Interpretation: Let us consider a plane curve. Since T is a unit vector,

$$T(t) = \cos\alpha(t) i + \sin\alpha(t) j$$

where $\alpha(t)$ is the angle between the tangent vector and the positive x-axis (see the figure). From the previous equation, we get

$$T'(t) = -\sin\alpha(t)\alpha'(t) i + \cos\alpha(t)\alpha'(t) j = \alpha'(t) u(t)$$

where $u(t) = \cos(\alpha(t) + \frac{\pi}{2}) i + \sin(\alpha(t) + \frac{\pi}{2}) j$ which is a unit vector. When $\alpha'(t) > 0$, the angle is increasing and in this case $N(t) = u(t)$. Similarly, when $\alpha'(t) < 0$, the angle is decreasing and we have $N(t) = -u(t)$.



Curvature of a curve: Curvature is related to the rate of change of the unit tangent with respect to the arc length. The curvature of a curve is $\kappa = \| \frac{dT}{ds} \|$. Since $\frac{dT}{ds} = \frac{dT}{dt}\frac{dt}{ds} = \frac{T'(t)}{\| \frac{dR}{dt} \|}$, the curvature is given by the following formula:

$$\kappa(t) = \frac{\| T'(t) \|}{\| \frac{dR}{dt} \|}.$$

Example 1: Suppose C is a circle of radius a defined by $R(t) = a\cos ti + a\sin tj$. This implies that $R'(t) = -a\sin ti + a\cos tj$, $T(t) = -\sin ti + \cos tj$ and $T'(t) = -\cos ti - \sin tj$. Since $\| R'(t) \| = a$ and $\| T'(t) \| = 1$, we have $\kappa = \frac{1}{a}$. So the circle has the constant curvature and the curvature is the reciprocal of the radius of the circle.

Example 2: Sometimes the curvature of a plane curve is defined to be the rate of change of the angle between the tangent vector and the positive x-axis. We will see that our definition coincides with this. For a plane curve, we have shown that $\| T'(t) \| = | \alpha'(t) |$, when $T(t) = \cos\alpha(t)i + \sin\alpha(t)j$. By the chain rule, $\frac{d\alpha}{dt} = \frac{d\alpha}{ds}\frac{ds}{dt} = \| \frac{dR}{dt} \| \frac{d\alpha}{ds}$. From the formula of curvature, we get

$$\kappa(t) = | \frac{d\alpha}{ds} | .$$

The formula given in the following theorem provides a simpler method for determining the curvature.

Theorem: Let $v(t)$ and $a(t)$ denote the velocity and the acceleration vectors of a motion of a particle on a curve defined by $R(t)$. Then

$$\kappa(t) = \frac{\| a(t) \times v(t) \|}{\| v(t) \|^3}.$$

We will not prove the previous theorem but we will use it.

Problem 1: If a plane curve has the Cartesian equation $y = f(x)$ where f is a twice differentiable function, then show that the curvature at the point $(x, f(x))$ is $\dfrac{|f''(x)|}{[1 + f'(x)^2]^{3/2}}$.

Solution: The graph of f can be considered as a parametric curve $R(t) = ti + f(t)j$. Then $v(t) = R'(t) = i + f'(x)j$ and $a(t) = R''(t) = f''(t)j$. This implies that $v(t) \times a(t) = f''(t)k$. Therefore, $\| v(t) \times a(t) \| = | f''(t) |$ and $\| v(t) \| = \sqrt{1 + f'(t)^2}$. Substituting these values in the formula of $\kappa(t)$, we get the final expression.

Problem 2: For the curve $R(t) = t\vec{i} + t^2\vec{j} + \frac{2}{3}t^3\vec{k}$ find the equations of the tangent, principal normal and binormal at $t = 1$. Also calculate the curvature of the curve.

Solution : Differentiating $R(t)$ we get, $R'(t) = i + 2tj + 2t^2k$. The unit tangent vector is given by

$$T(t) = \frac{R'(t)}{\| R'(t) \|} = \frac{i + 2tj + 2t^2k}{1 + 2t^2}.$$

Differentiating, we get

$$T'(t) = \frac{-4ti + (2 - 4t^2)j + 4tk}{(1 + 2t^2)^2}$$

and this is the direction of the normal. At $t = 1$, the unit tangent vector is $T = \frac{i+2j+2k}{3}$ and a normal vector is $\frac{-4i-2j+4k}{9}$. Therefore, the equation of the tangent is : $(x, y, z) = (1, 1, \frac{2}{3}) + t(1, 2, 2)$ and the equation of the principal normal is : $(x, y, z) = (1, 1, \frac{2}{3}) + t(-2, -1, 2)$. The direction of the binormal is defined to $b = T \times N$. Simple calculation will lead to the equation of the binormal.

To evaluate the curvature we can use the formula $\kappa(t) = \frac{\|T'(t)\|}{\|\frac{dR}{dt}\|}$ as we have already evaluated $T'(t)$ or the formula given in the previous theorem. In any way, we get $\kappa(t) = \frac{2}{(1+2t^2)^2}$.