Lecture 31 : Lagrange Multiplier Method

Let $f : S \to \mathbb{R}$, $S \subset \mathbb{R}^3$ and $X_0 \in S$. If X_0 is an interior point of the constrained set S, then we can use the necessary and sufficient conditions (first and second derivative tests) studied in the previous lecture in order to determine whether the point is a local maximum or minimum (i.e., local extremum) of f on S. If X_0 is not an interior point then one cannot apply these tests. For example, one cannot apply these tests at a point on a sphere $x^2 + y^2 + z^2 = c^2$, because no point is an interior point in this constrained set.

In general, constrained extremum problems are very difficult to solve and there is no general method for solving such problems. In case the constrained set is a level surface, for example a sphere, there is a special method called Lagrange multiplier method for solving such problems. So, we will be dealing with the following type of problem.

Problem : Find the local or absolute maxima and minima of a function $f(x, y, z)$ on the (level) surface $S := \{(x, y, z) : g(x, y, z) = 0\}$ where $g : \mathbb{R}^3 \to \mathbb{R}$.

Let us illustrate the problem with an example.

Example : Find a point on the plane $\{(x, y, z) : 2x + 3y - z = 5\}$ which is nearest to the origin of $\mathbb{R}^3$. Note that here we are minimizing the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ or $x^2 + y^2 + z^2$ over the constrained set S defined by $g(x, y, z) = 2x + 3y - z - 5$.

Necessary condition: Let $P_0 = (x_0, y_0, z_0)$ be a point on $S := \{(x, y, z) : g(x, y, z) = 0\}$. Suppose P_0 is a local extremum of f over S. Let us try to find a necessary condition. Our argument is going to be geometric and so we will not question certain assumptions made in the following argument.

Let us first assume that f and g have continuous partial derivatives and $\nabla g|_{P_0} \neq 0$. Let $C = R(t) = x(t)i + y(t)j + z(t)k$ be a curve on S passing through P_0 and let $P_0 = R(t_0)$. Since f has a local extremum at P_0 on S, it has a local extremum on C as well. Therefore $\frac{df}{dt}|_{t_0} = 0$. By the chain rule, $\nabla f \cdot \frac{dR}{dt} = 0$. Since the curve C is arbitrary, we conclude that $\nabla f|_{P_0}$ is perpendicular to the tangent plane of S at P_0. But we already know that $\nabla g|_{P_0}$ is also perpendicular to the tangent plane of S at P_0. Therefore, there exists $\lambda \in \mathbb{R}$ such that $\nabla f|_{P_0} = \lambda \nabla g|_{P_0}$.

So the following method is anticipated.

Lagrange Multiplier Method: Suppose f and g have continuous partial derivatives. Let $(x_0, y_0, z_0) \in S := \{(x, y, z) : g(x, y, z) = 0\}$ and $\nabla g(x_0, y_0, z_0) \neq 0$. If f has a local maximum or minimum at (x_0, y_0, z_0) then there exists $\lambda \in \mathbb{R}$ such that

$$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0).$$

To find the extremum points, in practice, we consider the following equations

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \;\text{ and }\; g(x, y, z) = 0. \tag{1}$$

These equations are solved for the unknowns x, y, z and λ. Then the local extremum points are found among the solutions of these equations.

Let us illustrate the method with a few examples.

Examples: 1. Let us find a point on the plane $2x + 3y - z = 5$ in $\mathbb{R}^3$ which is nearest to the origin. We have to minimize the function $f(x, y, z) = x^2 + y^2 + z^2$ subject to the constraint $g(x, y, z) = 2x + 3y - z - 5 = 0$. Here note that $\nabla g \neq 0$ at all points. The equations given in (1) imply that

$$2x = 2\lambda,\; 2y = 3\lambda,\; 2z = -\lambda \text{ and } 2x + 3y - z - 5 = 0.$$

Substituting $x = \lambda$, $y = \frac{3\lambda}{2}$ and $z = -\frac{\lambda}{2}$ in the equation $2x + 3y - z - 5 = 0$, we obtain that $\lambda = \frac{5}{7}$ and hence $\lambda = \frac{5}{7}$ and $(x, y, z) = (\frac{5}{7}, \frac{15}{14}, -\frac{5}{14})$ satisfy the equation (1). Since f attains its minimum on the plane, by the Lagrange multipliers method, the point $(\frac{5}{7}, \frac{15}{14}, -\frac{5}{14})$ has to be the nearest point.

2. Consider the problem of minimizing the function $f(x, y) = x^2 + y^2$ subject to the condition $g(x, y) = (x - 1)^3 - y^2 = 0$. The problem is to find a point on the curve $y^2 = (x - 1)^3$ which is nearest to the origin of $\mathbb{R}^2$. Geometrically, it is clear that the point $(1, 0)$ is the nearest point. But $\nabla g(1, 0) = 0$ while $\nabla f(1, 0) = (2, 0)$. Therefore $\nabla f(1, 0) \neq \lambda \nabla g(1, 0)$ for any λ. This explains that the condition $\nabla g(x_0, y_0, z_0) \neq 0$ cannot be dropped from the Lagrange multiplier method and a point at which ∇g is $(0, 0)$ could be an extremum point.

3. Let us evaluate the minimum and maximum value of the function $f(x, y) = 2 - x^2 - 2y^2$ subject to the condition $g(x, y) = x^2 + y^2 - 1$. If we use the Lagrange multiplier method, the equations in (1) imply that $2x + 2\lambda x = 0$, $4y + 2\lambda y = 0$ and $x^2 + y^2 - 1 = 0$. From the first two equations, we must have either $\lambda = -1$ or $\lambda = -2$. If $\lambda = -1$, then $y = 0$, $x = \pm 1$ and $f(x, y) = 1$. Similarly, if $\lambda = -2$, then $y = \pm 1$, $x = 0$ and $f(x, y) = 0$. Since the continuous function $f(x, y)$ achieves its maximum and minimum over the closed and bounded set $x^2 + y^2 = 1$, the points $(0, \pm 1)$ are the minima and $(\pm 1, 0)$ are the maxima, and the maximum value and the minimum value of f are 1 and 0 respectively.

This problem can also be solved using substitution as follows. Since $x^2 + y^2 - 1 = 0$, $x^2 = 1 - y^2$. Substituting this into f, we get $f(x, y) = 1 - y^2$. We are back to a one variable problem, which has a maximum at $y = 0$, where $x = \pm 1$ and $f(x, y) = 1$. Since $y \in [-1, 1]$, f has a minimum at $y = \pm 1$ where $x = 0$ and $f(x, y) = 0$. Although we solved this problem easily using substitution, it is usually very hard to solve such constrained problems using substitution.

4. Given n positive numbers $a_1, a_2, .., a_n$, let us find the maximum value of the expression $a_1 x_1 + a_2 x_2 + ... + a_n x_n$ where $x_1^2 + ... + x_n^2 = 1$. Note that here $f(x_1, x_2, .., x_n) = a_1 x_1 + a_2 x_2 + ... + a_n x_n$ and $g(x_1, x_2, ..., x_n) = x_1^2 + ... + x_n^2 - 1$. Although we stated the Lagrange multiplier method in $\mathbb{R}^3$, it works in $\mathbb{R}^n$ as well. The equations in (1) imply that $a_1 = 2\lambda x_1, ..., a_n = 2\lambda x_n$ and $x_1^2 + ... + x_n^2 - 1 = 0$. Therefore, $a_1^2 + a_2^2 + .. + a_n^2 = 4\lambda^2$. This implies that $\lambda = \pm \frac{\sqrt{a_1^2 + .. + a_n^2}}{2}$. Since the continuous function f achieves its minimum and maximum on the closed and bounded subset $x_1^2 + ... + x_n^2 = 1$, $\lambda = \frac{\sqrt{a_1^2 + .. + a_n^2}}{2}$ leads to the maximum value $f(\frac{a_1}{2\lambda}, \frac{a_2}{2\lambda}, ..., \frac{a_n}{2\lambda}) = \sqrt{a_1^2 + .. + a_n^2}$ and $\lambda = -\frac{\sqrt{a_1^2 + .. + a_n^2}}{2}$ leads to the minimum value of f.

Problem 1: Assume that among all rectangular boxes with fixed surface area of 20 square meters, there is a box of largest possible volume. Find its dimensions.

Solution: Let the box have sides of length x, y, $z > 0$. Then $V(x, y, z) = xyz$ and $xy + yz + xz = 10$. Using the method of Lagrange multipliers, we see that $yz = \lambda(y+z)$, $xz = \lambda(x+z)$ and $xy = \lambda(x+y)$. It is easy to see that $x, y, z > 0$. Now, we can see that $x = y = z$ and therefore, $x = y = z = \sqrt{\frac{10}{3}}$.

Problem 2: A company produces steel boxes at three different plants in amounts x, y and z, respectively, producing an annual revenue of $f(x, y, z) = 8xyz^2 - 200(x + y + z)$. The company is to produce 100 units annually. How should the production be distributed to maximize revenue?

Solution: Here, $g(x, y, z) = x + y + z - 100$. The Lagrange multiplier method implies that $8yz^2 - 200 = \lambda, 8xz^2 - 200 = \lambda, 16xyz - 200 = \lambda$ and $x + y + z - 100 = 0$. These imply that $x = y, z = 2x$ and $x = 25$.