Lecture 32 : Double integrals

In one variable calculus we had seen that the integral of a nonnegative function is the area under the graph. The double integral of a nonnegative function $f(x,y)$ defined on a region in the plane is associated with the volume of the region under the graph of $f(x,y)$.

The definition of double integral is similar to the definition of Riemannn integral of a single variable function. Let $Q=[a,b]\times[c,d]$ and $f:Q\to\mathbb{R}$ be bounded. Let P_1 and P_2 be partitions of $[a,b]$ and $[c,d]$ respectively. Suppose $P_1=\{x_0,x_1,...,x_n\}$ and $P_2=\{y_0,y_1,...,y_n\}$. Note that the partition $P=P_1\times P_2$ decomposes Q into mn sub-rectangles. Define $m_{ij}=\inf\{f(x,y):(x,y)\in[x_{i-1},x_i]\times[y_{j-1},y_j]\}$ and $L(P,f)=\sum_{i=1}^n\sum_{j=1}^m m_{ij}\triangle y_j\triangle x_i$. Similarly we can define $U(P,f)$. Define lower integral and upper integral as we do in the single variable case. We say that $f(x,y)$ is integrable if both lower and upper integral of $f(x,y)$ are equal. If the function $f(x,y)$ is integrable on Q then the double integral is denoted by

$$\iint\limits_Q f(x,y)dxdy \quad\text{or}\quad \iint\limits_Q f(x,y)dA.$$

The proof of the following theorem is similar to the single variable case.

Theorem: If a function $f(x,y)$ is continuous on a rectangle $Q=[a,b]\times[c,d]$ then f is integrable on Q.

Fubini's Theorem: In one variable case, we use the second FTC for calculating integrals. The following result, called Fubini's theorem, provides a method for calculating double integrals. Basically, it converts a double integral into two successive one dimensional integrations.

Theorem 32.1: Let $f:Q=[a,b]\times[c,d]\to\mathbb{R}$ be continuous. Then

$$\iint\limits_Q f(x,y)dxdy=\int_c^d\left(\int_a^b f(x,y)dx\right)dy=\int_a^b\left(\int_c^d f(x,y)dy\right)dx.$$

We will not present the proof of the previous theorem, instead we present a geometric interpretation of it.

Geometric interpretation: Let $f(x,y)>0$ for every $(x,y)\in Q$ and f be continuous. Consider the solid S enclosed by Q, the planes $x=a,\ x=b,\ y=c,\ y=d$ and the surface $z=f(x,y)$. From the way we have defined the double integral, we can consider the value $\iint\limits_Q f(x,y)dxdy$ as the volume of S. We will now use the method of slicing and calculate the volume of S.

For every $y\in[c,d]$, $A(y)=\int_a^b f(x,y)dx$ is the area of the cross section of the solid S cut by a plane parallel to the xz-plane. Therefore, it follows from the method of slicing that

$$\int_c^d\left(\int_a^b f(x,y)dx\right)dy=\int_c^d A(y)dy$$

is the volume of the solid S. The other two successive single integrals compute the volume of S by integrating the area of the cross section cut by the planes parallel to the yz-plane.

Double integral over general bounded regions: We defined the double integral of a function which is defined over a rectangle. We will now extend the concept to more general bounded regions.

Let $fx,y)$ be a bounded function defined on a bounded region D in the plane. Let Q be a rectangle such that $D\subseteq Q$. Define a new function $\widetilde{f}(x,y)$ on Q as follows:

$$\widetilde{f}(x,y)=f(x,y)\text{ if }(x,y)\in D \quad\text{and}\quad \widetilde{f}(x,y)=0\text{ if }(x,y)\in Q\setminus D.$$

Basically we have extended the definition of f to Q by making the function value equal to 0 outside D. If $\widetilde{f}(x,y)$ is integrable over Q, then we say that $f(x,y)$ is integrable over D and we define $\iint\limits_D f(x,y)dxdy = \iint\limits_Q \widetilde{f}(x,y)dxdy$. We find that defining the concept of double integral over a more general region D is a trivial one, but the important question is how to evaluate $\iint\limits_D f(x,y)$.

If D is a general bounded domain, then there is no general method to evaluate the double integral. However, if the domain is in a simpler form (as given in the following result) then there is a result to convert the double integral in to two successive single integrals.

Fubini's theorem (stronger form) : Let $f(x,y)$ be a bounded function over a region D.

1. If $D = \{(x,y) : a \leq x \leq b \text{ and } f_1(x) \leq y \leq f_2(x)\}$ for some continuous functions $f_1, f_2 : [a,b] \to \mathbb{R}$, then $\iint\limits_D f(x,y)dxdy = \int\limits_a^b \left(\int\limits_{f_1(x)}^{f_2(x)} f(x,y)dy \right) dx$.
2. If $D = \{(x,y) : c \leq y \leq d \text{ and } g_1(y) \leq x \leq g_2(y)\}$ for some continuous functions $g_1, g_2 : [c,d] \to \mathbb{R}$, then $\iint\limits_D f(x,y)dxdy = \int\limits_c^d \left(\int\limits_{g_1(y)}^{g_2(y)} f(x,y)dx \right) dy$.

If we use the method of slicing, as we did earlier, we can get the geometric interpretation of the previous theorem. Let us illustrate the method given in the previous theorem with some examples.

Example 1: Let us evaluate the integral $\iint\limits_D (x+y)^2 dxdy$ where D is the region bounded by the lines joining the points $(0,0)$, $(0,1)$ and $(2,2)$. Note that the domain D (see Figure 1) is the form given in the first part of the previous theorem with $a = 0, b = 2, f_1(x) = x$ and $f_2(x) = \frac{x}{2} + 1$. Therefore, by the previous theorem $\iint\limits_D (x+y)^2 dxdy = \int\limits_0^2 \left(\int\limits_x^{\frac{x}{2}+1} (x+y)^2 dy \right) dx$.

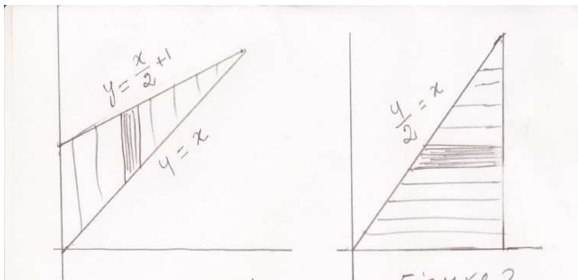

Figure 1 Figure 2

Example 2: Let us evaluate $\int\limits_0^2 \left(\int\limits_{\frac{y}{2}}^1 e^{x^2} dx \right) dy$. Note that we are given two consecutive single integrals. First we have to integrate w.r.to x and then w.r.to y. If we directly integrate then the calculation becomes complicated. So we will use Fubini's theorem and change the order of integration (i.e., $dxdy$ to $dydx$). Note that when we change the order of integration the limits will change.

We will first use Fubini's theorem and convert the consecutive single integrals in to a double integral over a domain D. Note that the integrals are of the form given in the second part of the previous theorem. By the previous theorem (going from right to left) we have $\int_0^2 \left(\int_{y/2}^1 e^{x^2} dx \right) dy = \iint_{D_2} f(x,y)dxdy$ where $D_2 = \{(x,y) : 0 \leq y \leq 2 \text{ and } \frac{y}{2} \leq x \leq 1\}$ (see Figure 2). Now we will use the first part of the previous theorem and convert this double integral into two consecutive single integrals. By the first part of the previous theorem, $\iint\limits_{D_2} f(x,y)dxdy = \int\limits_0^1 \left(\int\limits_0^{2x} e^{x^2} dy \right) dx = e - 1$.