Lecture 33 : Change of Variable in a Double Integral; Triple Integral

We used Fubini's theorem for calculating the double integrals. We have also noticed that Fubini's theorem can be applied if the domain is in a particular form. In this lecture, we will see that in some cases even if the domain is not in that particular form, using some change of variables, we can transform the original double integral into another double integral over a new region where we can apply Fubini's theorem. This idea is analogous to the method of substitution in single variable:

$$\int_a^b f(x)dx = \int_c^d f[g(t)]g'(t)dt, \tag{1}$$

where $a = g(c)$ and $b = g(d)$.

Change of Variable: Here we deal with the problem of transforming an integral $\iint_S f(x,y)dxdy$ defined over a region S in the xy-plane, into another integral $\iint_T F(x,y)dudv$ defined over a new region T in the uv-plane. Instead of the one function g given in (1), here we have two functions X and Y connecting x, y with u, v as follows: $x = X(u,v)$ and $y = Y(u,v)$. Note that a set of points in the uv-plane is mapped into another set of points in the xy-plane by the maps defined in the above equations.

Basic assumptions: We assume that the mapping from the domain T in the uv-plane to the domain S in the xy-plane is one-one. The functions X and Y are continuous and have continuous partial derivatives $\frac{\partial X}{\partial u}, \frac{\partial X}{\partial v}, \frac{\partial y}{\partial u}$ and $\frac{\partial Y}{\partial v}$. The Jacobian $J(u,v)$ defined below is never zero. The Jacobian is defined as follows:

$$J(u,v) = \begin{vmatrix} \frac{\partial X}{\partial u} & \frac{\partial Y}{\partial u} \\ \frac{\partial X}{\partial v} & \frac{\partial Y}{\partial v} \end{vmatrix}$$

The formula: $\iint\limits_S f(x,y)dxdy = \iint\limits_T f[X(u,v), Y(u,v)] \ \mid J(u,v) \mid dudv$

We note that the Jacobian defined in the previous equation and the function $g'(t)$ defined in (1) play similar roles in their respective equations. The proof of the change of variable formula is not easy and so we will not present it here.

Example: Let us find the area of the region S bounded by the hyperbolas $xy = 1$ and $xy = 2$, and the curves $xy^2 = 3$ and $xy^2 = 4$. Note that the area of S is $\iint_S dxdy$. Put $u = xy$ and $v = xy^2$, then $x = \frac{u^2}{v}$ and $y = \frac{v}{u}$. The region T is: $1 \le u \le 2$ and $3 \le v \le 4$. The Jacobian $J(u,v) = \frac{1}{v}$. Therefore by the change of variable formula the area of $S = \iint_S dxdy = \iint_T \frac{1}{v}dudv = \int\limits_3^4 \int\limits_1^2 \frac{1}{v}dudv$.

Special case (Polar coordinate): In this case the variables x and y are changed to r and θ by the following two equations: $x = X(r,\theta) = r\cos\theta$ and $y = Y(r,\theta) = r\sin\theta$. We assume that $r > 0$ and θ lies in $[0, 2\pi)$ or $\theta_0 \le \theta < \theta_0 + 2\pi$ for some θ_0 so that the mapping involved in the change of variable is one-one. The Jacobian is

$$J(u,v) = \begin{vmatrix} \frac{\partial X}{\partial r} & \frac{\partial Y}{\partial r} \\ \frac{\partial X}{\partial \theta} & \frac{\partial Y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r(\cos^2\theta + \sin^2\theta) = r.$$

Hence the change of variable formula in this case is : $\iint\limits_S f(x,y)dxdy = \iint\limits_T f(r\cos\theta, r\sin\theta)\ rdrd\theta$.

Example: Let us find the volume of the sphere of radius a. The volume is

$$V = 2\iint\limits_S \sqrt{a^2 - x^2 - y^2}dxdy \quad \text{where} \quad S = \{(x,y) : x^2 + y^2 \le a^2\}.$$

If we use the rectangular coordinates, $V = 4\int\limits_{-a}^{a} \int\limits_{0}^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}dydx$ which is complicated to calculate. Let use use the polar coordinates. In polar coordinate

$$V = 2\iint\limits_{T} \sqrt{a^2-r^2}\,r dr d\theta \quad \text{where} \quad T = [0,a]\times[0,2\pi].$$

By Fubini's theorem, $V = 2\int\limits_0^a\int\limits_0^{2\pi} \sqrt{a^2-r^2}\,r d\theta dr = 4\pi \int\limits_0^a r\sqrt{a^2-r^2}dr = 4\pi \frac{(a^2-r^2)^{\frac{3}{2}}}{-3} \Big|_0^a = \frac{4\pi a^3}{3}$.

Triple integrals: In the previous lecture, we extended the concept of integrals for functions defined on $[a,b]\times[c,d]$. The same can be extended to functions defined on $Q = [a,b]\times[c,d]\times[e,f]$. The definition of integral of such a function is entirely analogous to the definition of double integrals. Every partition P of Q is of the form $P = P_1\times P_2\times P_3$ where P_1, P_2 and P_3 are partitions of $[a,b],[b,c]$ and $[e,f]$ respectively. For a given partition P and a bounded function defined on Q we can define $L(P,f), U(P,f)$, lower integral, upper integral and integral of f as we defined in the double integral case. If a function f on Q is integrable then the integral, called triple integral, is denoted by

$$\iiint_Q f(x,y,z)dxdydz \quad \text{or} \quad \iiint_Q f(x,y,z)dV.$$

As we did in the double integral case, the definition of triple integral can be extended to any bounded region in $\mathbb{R}^3$. One can also prove that every continuous function on Q is integrable.

Remark: In the double integral case, the integral of positive function f is the volume of the region below the surface $z = f(x,y)$. In the triple integral case we do not have any such geometric interpretation, except the fact that $\iiint_D dxdydz$ is considered to be the volume of the region D. The concept of double integrals can be used in applications, for example, to define the center of mass and moments of inertia of a two dimensional object (see the text book). Similarly, the triple integrals are used in applications which we are not going to see. In this lecture we will see how to evaluate the triple integrals.

There is a Fubini's theorem to evaluate the triple integrals.

Fubini's theorem: Let D be a bounded domain in $\mathbb{R}^3$ described as follows:

$$D = \{(x,y,z) : (x,y)\in R \quad \text{and} \quad f_1(x,y)\le z\le f_2(x,y)\}.$$

That is, D is bounded above by the surface $z = f_1(x,y)$, bounded below by the surface $z = f_2(x,y)$ and on the side by the cylinder generated by a line moving parallel to the z-axis along the boundary of R. The projection of D on the xy-plane is the region R. For example, consider

$$D = \{(x,y,z) : 0\le x^2+y^2\le 2,\ x^2+y^2\le z\le 2\}.$$

Here R is a circular region and D is bounded below by the paraboloid $z = x^2+y^2$ and above by the plane $z = 2$. The following theorem converts a triple integral into iterated integrals of one and two dimensions.

Theorem: If f is continuous on D and f_1, f_2 are continuous on R then

$$\iiint_D f(x,y,z)dV = \iint_R \left(\int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)dz\right) dA.$$

Example: Let us compute $\iiint_D xdxdydz$ where D is the region in space bounded by $x = 0,\ y = 0,\ z = 2$ and the surface $z = x^2+y^2$. Note that $D = \{(x,y,z) : (x,y)\in R,\ x^2+y^2\le z\le 2\}$ where $R = \{(x,y) : 0\le x\le\sqrt{2},\ 0\le y\le\sqrt{2-x^2}\}$. Therefore by the previous theorem,

$$\iiint_D xdxdydz = \iint_R \left(\int_{x^2+y^2}^{2} xdz\right) dA = \int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_{x^2+y^2}^{2} x\ dzdydx = \frac{8\sqrt{2}}{15}.$$