Lecture 35 : Surface Area; Surface Integrals

In the previous lecture we defined the surface area $a(S)$ of the parametric surface S, defined by $r(u,v)$ on T, by the double integral

$$a(S) = \iint_T \| r_u \times r_v \| \, dudv. \tag{1}$$

We will now drive a formula for the area of a surface defined by the graph of a function.

Area of a surface defined by a graph: Suppose a surface S is given by $z = f(x,y),\ (x,y) \in T$, that is, S is the graph of the function $f(x,y)$. (For example, S is the unit hemisphere defined by $z = \sqrt{1 - x^2 - y^2}$ where (x,y) lies in the circular region $T : x^2 + y^2 \leq 1$.) Then S can be considered as a parametric surface defined by:

$$r(x,y) = xi + yj + f(x,y)k, \quad (x,y) \in T.$$

In this case the surface area becomes

$$a(S) = \iint_T \sqrt{1 + f_x^2 + f_y^2}\ dxdy. \tag{2}$$

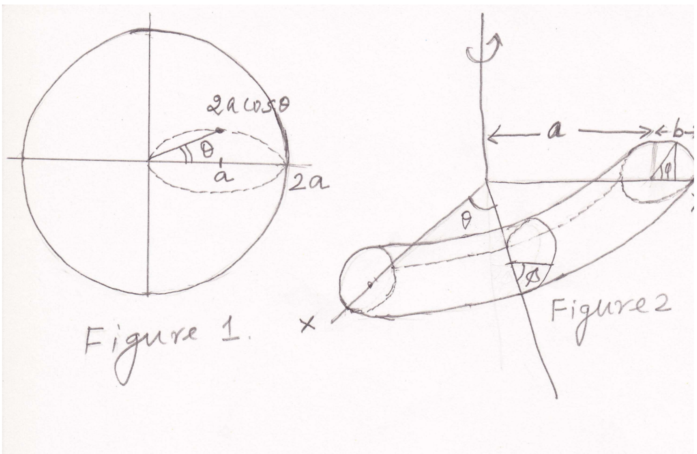
because $\| r_u \times r_v \| = \| -f_x i - f_y j + k \| = \sqrt{1 + f_x^2 + f_y^2}$.

Example 1: Let us find the area of the surface of the portion of the sphere $x^2 + y^2 + z^2 = 4a^2$ that lies inside the cylinder $x^2 + y^2 = 2ax$. Note that the sphere can be considered as a union of two graphs: $z = \pm\sqrt{4a^2 - x^2 - y^2}$. We will use the formula given in (2) to evaluate the surface area. Let $z = f(x,y) = \sqrt{4a^2 - x^2 - y^2}$. Then

$$f_x = \frac{-x}{\sqrt{4a^2 - x^2 - y^2}}, \quad f_y = \frac{-y}{\sqrt{4a^2 - x^2 - y^2}} \quad \text{and} \quad \sqrt{1 + f_x^2 + f_y^2} = \sqrt{\frac{4a^2}{4a^2 - x^2 - y^2}}.$$

Let T be the projection of the surface $z = f(x,y)$ on the xy-plane (see Figure 1). Then, because of the symmetry, the surface area is

$$a(S) = 2\iint_T \sqrt{\frac{4a^2}{4a^2 - x^2 - y^2}} dxdy = 2 \times 2 \int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} \frac{2ardrd\theta}{\sqrt{4a^2 - r^2}}.$$



Remark: Since

$$\| r_u \times r_v \|^2 = \| r_u \|^2 \| r_v \|^2 \sin^2\theta \ = \| r_u \|^2 \| r_v \|^2 \,(1 - \cos^2\theta) \ = \| r_u \|^2 \| r_v \|^2 - (r_u \cdot r_v)^2,$$

the formula given in (1) can be written as

$$a(S) = \iint_T \sqrt{EG - F^2} dudv \tag{3}$$

where $E = r_u \cdot r_u, G = r_v \cdot r_v$ and $F = r_u \cdot r_v$.

Example 2: Let us compute the area of the torus

$$x = (a + b\cos\phi)\cos\theta,\ y = (a + b\cos\phi)\sin\theta,\ z = b\sin\phi$$

where $0 \le \theta \le 2\pi,\ 0 \le \phi \le 2\pi$, and a and b are constants such that $0 < b < a$. Since the surface is given in the parametric form with the parameters θ and ϕ, we can either use the formula given in (1) or (3) and find the surface area. We do not have to know how the surface looks like. However the surface is given in Figure 2 for understanding. Note that

$$r_\theta = -(a + b\cos\phi)\sin\theta i + (a + b\cos\phi)\cos\theta j + 0k, \quad r_v = -b\sin\phi\cos\theta i - b\sin\phi\sin\theta j + b\cos\phi k.$$

This implies that $E = r_u \cdot r_u = (a+b\cos\phi)^2,\ F = 0, G = b^2$ and hence $\sqrt{EG - F^2} = b(a+b\cos\varphi)$. Therefore, by (3), the surface area is

$$a(S) = \iint\limits_T b(a + b\cos\varphi)d\theta d\phi = \int_0^{2\pi}\int_0^{2\pi} b(a + b\cos\varphi)d\theta d\phi = 4\pi^2 ab.$$

Note that this problem can also be solved using the Pappus theorem : $a(S) = 2\pi\rho L = 2\pi \cdot a \cdot 2\pi b$.

Surface Integrals: We will define the concept of integrals, called surface integrals, to the scalar functions defined on parametric surfaces. Surface integrals are used to define center of mass and moment of inertia of surfaces, and the surface integrals occur in several applications. We will not get in to the applications of the surface integrals in this course. We will define the surface integrals and see how to evaluate them.

Let S be a parametric surface defined by $r(u, v), (u, v) \in T$. Suppose r_u and r_v are continuous. Let $g : S \to \mathbb{R}$ be bounded. The surface integral of g over S, denoted by $\iint\limits_S g d\sigma$, is defined by

$$\iint\limits_S g\ d\sigma = \iint\limits_T g(r(u, v)) \parallel r_u \times r_v \parallel dudv = \iint\limits_T g(r(u, v))\sqrt{EG - F^2}\ dudv \tag{4}$$

provided the RHS double integral exists. If S is defined by $z = f(x, y)$, then

$$\iint\limits_S g\ d\sigma = \iint\limits_T g[x, y, f(x, y)]\sqrt{1 + f_x^2 + f_y^2}\ dxdy. \tag{5}$$

where T is the projection of the surface S over the xy-plane.

Example 3: Let S be the hemispherical surface $z = (a^2-x^2-y^2)^{1/2}$. Let us evaluate $\iint\limits_S \frac{d\sigma}{[x^2+y^2+(z+a)^2]^{1/2}}$. We first parameterize the surface S as follows:

$$S := r(\theta, \phi) = (a\sin\phi\cos\theta, \quad a\sin\phi\sin\theta,\ a\cos\phi), \quad 0 \le \theta \le 2\pi, \quad 0 \le \phi \le \pi.$$

Simple calculation shows that $\sqrt{EG - F^2} = a^2\sin\phi$ and $[x^2+y^2+(z+a)^2]^{1/2} = 2a\cos\frac{\phi}{2}$. Therefore, by equation (4), the surface integral is

$$\iint\limits_S \frac{d\sigma}{[x^2+y^2+(z+a)^2]^{1/2}} = \int_0^{2\pi}\int_0^{\pi/2} \frac{a^2\sin\phi}{2a\cos\frac{\phi}{2}}\ d\phi d\theta.$$

Example 4: Let us evaluate the surface integral $\iint_S g\ d\sigma$ where $g(x, y, z) = x+y+z$ and the surface S is described by $z = 2x+3y,\ x \ge 0,\ y \ge 0$ and $x+y \le 2$. We use the formula given in (5) to evaluate the surface integral. Note that the projection T of the surface is $\{(x, y) : x \ge 0,\ y \ge 0,\ x + y \le 2\}$. The surface integral is

$$\iint\limits_S g\ d\sigma = \iint\limits_T (x + y + z)\sqrt{1 + f_x^2 + f_y^2}\ dxdy = \int_0^2\int_0^{2-y}(x + y + 2x + 3y)\sqrt{14}\ dxdy.$$

Remark: Under certain general conditions (we deal with surfaces satisfying such conditions) the value of the surface integral is independent of the representation.