Lecture 36: Line Integrals; Green's Theorem

Let $R : [a, b] \to \mathbb{R}^3$ and C be a parametric curve defined by $R(t)$, that is $C(t) = \{R(t) : t \in [a, b]\}$. Suppose $f : C \to \mathbb{R}^3$ is a bounded function. In this lecture we define a concept of integral for the function f. Note that the integrand f is defined on $C \subset \mathbb{R}^3$ and it is a vector valued function. The integral of such a type is called a line integral or a contour integral.

Definition: Suppose R is a differentiable function. The line integral of f along C is denoted by the symbol $\int_C f \cdot dR$ and is defined by

$$\int_C f \cdot dR = \int_a^b f(R(t)) \cdot R'(t)dt$$

provided the RHS integral exists.

The line integrals appear in several physical situations in which the behavior of a vector is studied along a curve such as work done by a force over a curve, flux of the fluid's velocity vector across a curve and so on. We will not deal with such physical situations in this course, however, we will see that the line integrals are useful to calculate certain types of double integrals and areas of plane regions enclosed by parametric curves.

Suppose $f = (f_1, f_2, f_3)$ and $R(t) = (x(t), y(t), z(t))$ then the line integral $\int_C f \cdot dR$ is also written as $\int_C f_1dx + f_2dy + f_3dz$ or $\int_C f_1(x, y, z)dx + f_2(x, y, z)dy + f_3(x, y, z)dz$.

Example 1: Let us compute the line integral $\int_C f \cdot dR$ from $(0, 0, 0)$ to $(1, 2, 4)$ if $f = x^2i + yj + (xz - y)k$
(a) along the line segment joining these two points.
(b) along the curve given parametrically by $x = t^2,\ y = 2t,\ z = 4t^3$.

Solution: (a) Parameterize the line segment as follows: $x = t,\ y = 2t,\ z = 4t$. Then

$$\int_C f \cdot dR = \int_C x^2dx + ydy + (xz - y)dz = \int_0^1 t^2dt + (2t)(2dt) + (4t^2 - 2t)(4dt) = \int_0^1 (17t^2 - 4t)dt = \tfrac{11}{3}.$$

(b) The parametrization is already given. Repeat the steps given in the solution of (a).

Problem 1: Evaluate $\int\limits_C \frac{-ydx+xdy}{x^2+y^2}$, where $C := \{(x, y) : x^2 + y^2 = r^2\},\ r > 0$.

Solution: Let us consider $C = (rcost, rsint), 0 \leq t \leq 2\pi$. Then $\int\limits_C \frac{-ydx+xdy}{x^2+y^2} = \int\limits_0^{2\pi} \frac{sin^2t+cos^2t}{sin^2t+cos^2t}dt = 2\pi$

Remark: One can show that a line integral is independent of the parametrization (that preserves the orientation).

The second FTC for line integrals: The second FTC for real functions states that if $f : [a, b] \to \mathbb{R}$ and f' is continuous then $\int_a^b f'(t) = f(b) - f(a)$. This says that the value of the integral (of some function) depends only on the end points and not on the points between them. We will first extend this result to line integrals.

Theorem: Let $S \subset \mathbb{R}^3$, $f : S \to \mathbb{R}$ be differentiable on S and the gradient ∇f be continuous. Let A, B be two points in S. Let $C = \{R(t) : t \in [a, b]\}$ be a curve lying in S and joining the points A and B, that is $R(a) = A$ and $R(b) = B$. Suppose $R't)$ is continuous on $[a, b]$. Then

$$\int_C \nabla f \cdot dR = f(B) - f(A).$$

Proof: Let $g(t) = f(R(t)$. Then

$$\int_C \nabla f \cdot dR = \int_a^b \nabla f(R(t)) \cdot R'(t)dt = \int_a^b g'(t)dt = g(b) - g(a) = f(R(b)) - f(R(a)) = f(B) - f(A).\square$$

Example 2: Since $\int_C ydx + xdy = \int_C \nabla(xy) \cdot (dx, dy)$, by the previous theorem, the line integral is independent of path joining any two points.

Green's Theorem: The above theorem states that the line integral of a gradient is independent of the path joining two points A and B. Moreover, the line integral of a gradient along a path joining two points A and B is expressed in terms of the values of f at the boundary points A and B. This is analogous to the second FTC of real functions. We will now see a two dimensional analog of the second FTC theorem. It states that a double integral (of certain type of function) over a plane region R can be expressed as a line integral (of some function) along the boundary curve of R. This result is called Green's theorem. To present the formal statement of Green's theorem we need the following definitions.

Let $R : [a, b] \to \mathbb{R}^3$ be continuous.

Simple closed curve: If $R(a) = R(b)$ then the curve described by R is closed. A closed curve such that $R(t_1) \neq R(t_2)$ for every t_1, t_2 in $(a, b]$ is called a simple closed curve.

Piecewise smooth curve: If R' exists and continuous then the curve described by R is called smooth. The curve is called piecewise smooth if the interval $[a, b]$ can be partitioned into a finite number of subintervals and in each of which the curve is smooth.

Theorem: Let C be a piecewise smooth simple closed curve in the xy-plane and let D denote the closed region enclosed by C. Suppose $M, N, \frac{\partial N}{\partial x}$ and $\frac{\partial M}{\partial y}$ are real valued continuous functions in an open set containing D. Then

$$\iint\limits_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dxdy = \oint_C (Mi + Nj) \cdot dR = \oint_C Mdx + Ndy$$

where the line integral is taken around C in the counterclockwise direction.

Remark: In the above theorem we have made some casual statements such as "closed region enclosed by C" and "counterclockwise direction". These are intuitively evident, however, formal definitions and some explanations are required which we are not going to provide. We will also make a few such statements in the next two or three lectures. So we have to be aware that our treatment is not completely rigorous. Proof of the previous theorem for certain special regions is given in the text book and the proof in the general form is not easy.

An application. *Area expressed as a line integral:* Let C be a simple (piecewise smooth) closed curve and D be the region enclosed by C. Let $N(x, y) = \frac{x}{2}$ and $M(x, y) = -\frac{y}{2}$, then by Green's theorem the area of D is

$$a(D) = \iint\limits_D dxdy = \iint\limits_D (N_x - M_y)dxdy = \int_a^b Mdx + Ndy = \frac{1}{2} \int_C -ydx + xdy.$$

Examples: 1. Let us show that the value of $\int_C xy^2 dx + (x^2 y + 2x)dy$ around any square depends only on the size of the square C and not on its location in the plane. Let R be a square enclosed by the boundary C. By Green's theorem

$$\int\limits_C xy^2 dx + (x^2 y + 2x)dy = \iint\limits_R 2dxdy = 2 \text{ Area(R)}.$$

2. We will use the formula given above to find the area bounded by the ellipse $C : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Parametrize C by $(a \cos t, b \sin t),\ 0 \leq t \leq 2\pi$. Then the area is

$$\frac{1}{2} \int_C -ydx + xdy = \frac{1}{2} \int_0^{2\pi} -(b \sin t)(-a \sin t)dt + (a \cos t)(b \cos t)dt = \frac{1}{2} \int_0^{2\pi} abdt = ab\pi.$$