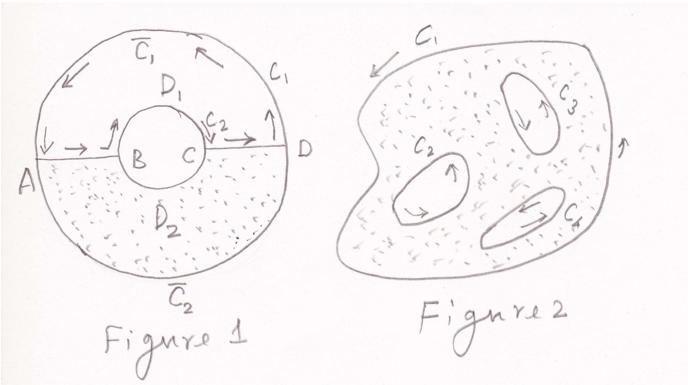
Lecture 37: Green's Theorem (contd.); Curl; Divergence

We stated Green's theorem for a region enclosed by a simple closed curve. We will see that Green's theorem can be generalized to apply to annular regions.

Suppose C_1 and C_2 are two circles as given in Figure 1. Consider the annular region (the region between the two circles) D. Introduce the crosscuts AB and CD as shown in Figure 1. Consider the simple closed curve $\overline{C_1}$ consisting of the upper half of C_2, the upper half of C_1, and the segments AB and CD as shown in Figure 1. Similarly, consider the simple closed curve $\overline{C_2}$ consisting of the lower half of C_2, the lower half of C_1, and the segments AB and CD. Let D_1 and D_2 be the regions enclosed by $\overline{C_1}$ and $\overline{C_2}$.



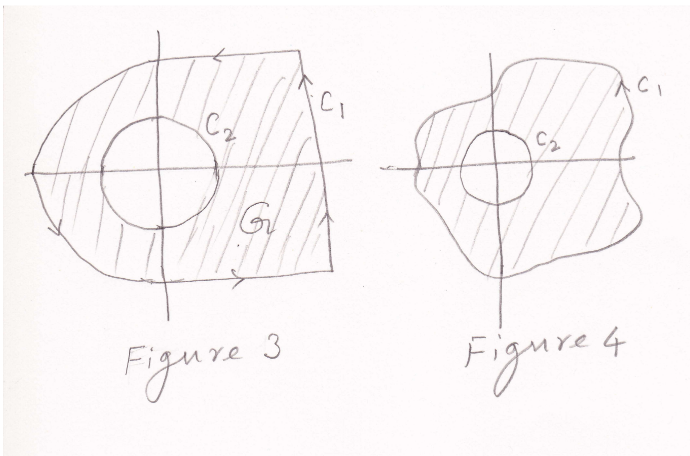
Suppose we are given two continuously differentiable scalar valued functions M and N on an open set containing the annular region D. Let us now apply Green's theorem to each of the regions D_1 and D_2 and add the two identities obtained from Green's theorem. Since the line integrals along the crosscuts cancel, we obtain

$$\iint\limits_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy = \oint_{C_1}(Mdx + Ndy) - \oint_{C_2}(Mdx + Ndy). \tag{1}$$

In the above equation, the line integrals are taken around the curves in the counterclockwise directions. Note that when we apply Green's theorem on D_1, the line integral on the part of $\overline{C_2}$ is taken along the clockwise direction. So a minus sign appears in the above equation.

We note that using the idea given above we can generalize Green's theorem to apply to regions enclosed by two or more simple closed curves similar to the one given in Figure 2.

Example 1: Let G be the region outside the unit circle which is bounded on left by the parabola $y^2 = 2(x+2)$ and on the right by the line $x = 2$. Use Green's theorem to evaluate $\int_{C_1} \frac{xdy - ydx}{x^2+y^2}$ where C_1 is the outer boundary of G oriented counterclockwise.



Solution: Let C_2 be the unit circle (see Figure 3). If we take $M = -\dfrac{y}{x^2+y^2}$ and $N = \dfrac{x}{x^2+y^2}$, then a simple calculation shows that $N_x - M_y = 0$. Therefore $\iint_G (N_x - M_y)dx\,dy = 0$. By applying Green's theorem on G (as we did above to obtain (1)), we get

$$\oint_{C_1}(Mdx + Ndy) = \oint_{C_2}(Mdx + Ndy)$$

where the line integrals around both the curves are taken in the counterclockwise directions. We have already seen in Problem 1 of the previous lecture that $\oint_{C_2}(Mdx + Ndy) = 2\pi$. □

Remark: If we take C_2 be any circle centered at $(0,0)$ and C_1 be any (piecewise smooth) simple closed curve such that C_2 lies in the interior of C_1 as shown in Figure 4, by repeating the argument given in the above solution, we can show that $\oint_{C_1}(Mdx + Ndy) = 2\pi$ where M and N are given in the previous example.

Problem: Evaluate $\int_C \frac{xdy - ydx}{x^2+y^2}$ along any simple closed curve C in the xy plane not passing through the origin. Distinguish the cases where the region D enclosed by C: (a) includes the origin (b) does not include the origin.

Solution: (a) First note that if we take $M = -\dfrac{y}{x^2+y^2}$ and $N = \dfrac{x}{x^2+y^2}$, then the functions are not defined in the region D, hence one cannot apply Green's theorem. Choose a circle C_r of radius r centered at $(0,0)$ and C_r lies in the interior of C. Now one can apply Green's theorem on the region between these two curves. By the above remark, the value of the line integral is 2π.

(b) In this region we can apply Green's theorem. Therefore $\int\limits_C Mdx + Ndy = \iint\limits_D (N_x - M_y)dx\,dy = 0$.

Curl and divergence: In the previous two lectures we discussed Green's theorem which expresses a double integral (of certain type of function) over a plane region D as a line integral over the boundary of D. We have also noted that this is a two dimensional analog of the second FTC. In the next two lectures we will see two generalizations of Green's theorem involving surface integrals and triple integrals. These results are known as Stokes theorem and divergence theorem respectively. They are also, essentially, analogs of the second FTC.

We first rewrite Green's theorem into two different forms involving the concepts curl and divergence and then generalize these forms to surface integrals and triple integrals. Let us define the concepts curl and divergence.

Let $F : \mathbb{R}^3 \to \mathbb{R}^3$ and $F(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k$. Such functions are called vector field.

Curl: The curl of F is another vector field denoted by $curlF$ and defined by

$$curlF = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) i + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) k.$$

We rewrite the curl as follows: $curlF = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \nabla \times f$. These expressions can be easily remembered. However, while expanding the determinant it is understood that $\frac{\partial}{\partial x}$ times Q is to be interpreted as $\frac{\partial Q}{\partial x}$ and the symbol ∇ has to be treated as if it is vector $\nabla = \frac{\partial}{\partial x}i + \frac{\partial}{\partial y}j + \frac{\partial}{\partial z}k$.

Divergence: The divergence of F is a scalar valued function denoted by $divF$ and is defined by

$$divF = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

We can rewrite the div F as follows : $divF = \nabla \cdot F$. Note that we interpret $\frac{\partial}{\partial x}$ times Q as $\frac{\partial Q}{\partial x}$.