Lecture 39: The Divergence Theorem

In the last few lectures we have been studying some results which relate an integral over a domain to another integral over the boundary of that domain. In this lecture we will study a result, called divergence theorem, which relates a triple integral to a surface integral where the surface is the boundary of the solid in which the triple integral is defined.

Divergence theorem is a direct extension of Green's theorem to solids in $\mathbb{R}^3$. We will now rewrite Green's theorem to a form which will be generalized to solids.

Let D be a plane region enclosed by a simple smooth closed curve C. Suppose $F(x,y) = M(x,y)i + N(x,y)j$ is such that M and N satisfy the conditions given in Green's theorem. If the curve C is defined by $R(t) = x(t)i + y(t)j$ then the vector $\mathbf{n} = \frac{dy}{ds}i - \frac{dx}{ds}j$ is a unit normal to the curve C because the vector $T = \frac{dx}{ds}i + \frac{dy}{ds}j$ is a unit tangent to the curve C. By Green's theorem

$$\oint_C (F \cdot \mathbf{n})ds = \oint_C Mdy - Ndx = \iint_D \left(\frac{\partial M}{\partial x} - (-\frac{\partial N}{\partial y})\right) dxdy = \iint_D \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dxdy.$$

Since $divF = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$, Green's theorem takes the following form:

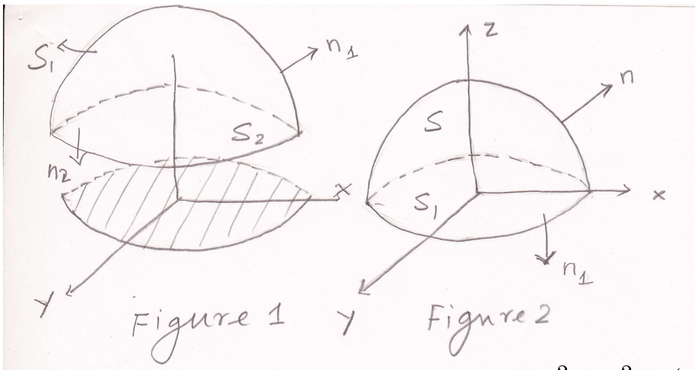
$$\iint_D divF dxdy = \oint_C (F \cdot \mathbf{n})ds.$$

We will now generalize this form of Green's theorem to a vector field F defined on a solid.

Theorem: Let D be a solid in $\mathbb{R}^3$ bounded by piecewise smooth (orientable) surface S. Let $F(x,y,z) = P(x,y,z)i + Q(x,y,z)j + R(x,y,z)k$ be a vector field such that P, Q and R are continuous and have continuous first partial derivatives in an open set containing D. Suppose $\mathbf{n}$ is the unit outward normal to the surface S. Then

$$\iiint_D divF \ dV \ = \ \iint_S F \cdot \mathbf{n} \ d\sigma.$$

Remark: The divergence theorem can be extended to a solid that can be partitioned into a finite number of solids of the type given in the theorem. For example, the theorem can be applied to a solid D between two concentric spheres as follows. Split D by a plane and apply the theorem to each piece and add the resulting identities as we did in Green's theorem.



Example: Let D be the region bounded by the hemispehere : $x^2 + y^2 + (z-1)^2 = 9, \ 1 \leq z \leq 4$ and the plane $z = 1$ (see Figure 1). Let $F(x,y,z) = xi + yj + (z-1)k$. Let us evaluate the integrals given in the divergence theorem.

Triple integral: Note that $divF = 3$. Therefore,

$$\iiint_D divF \ dV = \iiint_D 3 \ dV = 3 \cdot \frac{2}{3}\pi 3^3 = 54\pi.$$

Surface integral: The solid D is bounded by a surface S consisting of two smooth surfaces S_1 and S_2 (see Figure 1). Therefore

$$\iint_S F \cdot \mathbf{n}\ d\sigma = \iint_{S_1} F \cdot \mathbf{n}_1\ d\sigma + \iint_{S_2} F \cdot \mathbf{n}_2\ d\sigma.$$

Surface integral over the hemisphere S_1: The surface S_1 is given by:

$$g(x, y, z) = x^2 + y^2 + (z-1)^2 - 9 = 0.$$

An unit normal is

$$\mathbf{n}_1 = \frac{\nabla g}{\parallel \nabla f \parallel} = \frac{xi + yj + (z - 1k}{\sqrt{x^2 + y^2 + (z-1)^2}} = \frac{x}{3}i + \frac{y}{3}j + \frac{(z-1)}{3}k.$$

This is expected because the position vector is a normal to the sphere. It is clear that the normal obtained is the outward normal. This implies that over S_1,

$$F \cdot \mathbf{n}_1 = (x, y, z-1) \cdot (\frac{x}{3}, \frac{y}{3}, \frac{(z-1)}{3}) = 3.$$

Therefore

$$\iint\limits_{S_1} F \cdot n_1\ d\sigma = 3 \iint\limits_{S_1} d\sigma = 3 \cdot (surface\ area) = 3 \cdot 18\pi = 54\pi.$$

Surface integral over the plane region S_2: Here the outward normal $\mathbf{n}_2 = -k$. Therefore $F \cdot \mathbf{n}_2 = -z + 1$. Since on S_2, $z = 1$

$$\iint\limits_{S_2} F \cdot \mathbf{n}_2\ d\sigma = 0.$$

Hence $\iint_S F \cdot \mathbf{n}\ d\sigma = 54\pi$.

Problem: Use the divergence theorem to evaluate the surface integral $\iint_S F{\cdot}n\ d\sigma$ where $F(x, y, z) = (x + y, z^2, x^2)$ and S is the surface of the hemisphere $x^2 + y^2 + z^2 = 1$ with $z > 0$ and $\mathbf{n}$ is the outward normal to S.

Solution: First note that the surface is not closed. If we apply the divergence theorem to the solid $D := x^2 + y^2 + z^2 \le 1,\ z > 0$, we get

$$\iiint\limits_D divF\ dV \ = \iint\limits_S F \cdot \mathbf{n}\ d\sigma + \iint\limits_{S_1} F \cdot \mathbf{n}_1\ d\sigma$$

where $S_1 := x^2 + y^2 < 1,\ z = 0$ the base of the hemisphere (see Figure 2) and $\mathbf{n}_1$ is the outward normal to S_1 which is $-k$. Since $divF = 1$, the volume integral in the above equation is the volume of the hemisphere, $\frac{2\pi}{3}$. Therefore

$$\iint\limits_S F \cdot \mathbf{n}\ d\sigma = \frac{2\pi}{3} - \iint\limits_{S_1} F \cdot -k\ d\sigma = \frac{2\pi}{3} + \iint\limits_{S_1} x^2\ d\sigma$$

which is relatively easier to evaluate. To evaluate the surface integral over S_1, consider $S_1 = (\cos\theta, r\sin\theta),\ 0 \le r < 1,\ 0 \le \theta \le 2\pi$. Then

$$\iint\limits_{S_1} x^2\ d\sigma = \int\limits_0^1 \int\limits_0^{2\pi} r^2 \cos^2\theta r d\theta dr = \int\limits_0^1 r^3 \pi dr = \frac{\pi}{4}.$$

Therefore the requied integral is

$$\iint\limits_S F \cdot \mathbf{n}\ d\sigma = \frac{2\pi}{3} + \frac{\pi}{4} = \frac{11\pi}{12}.$$