Practice Problems 19 : Area between two curves, Polar coordinates

1. Find the area of the region enclosed by $y = \cos x,\ y = \sin x\ x = \frac{\pi}{2}$ and $x = 0$.

2. Consider the curves $y = x^3 - 9x$ and $y = 9 - x^2$.

   (a) Show that the curves intersect at $(-3, 0), (-1, 8)$ and $(3, 0)$.

   (b) Find the area of the region bounded by the curves.

3. Sketch the graphs of the following polar equations:

   (a) $r = \cos\theta$ (b) $r = -\cos\theta$ (c) $r = \sin\theta$ (d) $r = -\sin\theta$.

4. Sketch the limacons (convex or oval limacons, limacons with dimples, cardiods and limacons with inner loops).

   (a) $r = 3 + \cos\theta$ (b) $r = \frac{3}{2} + \cos\theta$ (c) $r = 1 + \cos\theta$ (d) $r = \frac{1}{2} + \cos\theta$
   (e) $r = 3 - \cos\theta$ (f) $r = \frac{3}{2} - \cos\theta$ (g) $r = 1 - \cos\theta$ (h) $r = \frac{1}{2} - \cos\theta$
   (i) $r = 3 + \sin\theta$ (j) $r = \frac{3}{2} + \sin\theta$ (k) $r = 1 + \sin\theta$ (l) $r = \frac{1}{2} + \sin\theta$
   (m) $r = 3 - \sin\theta$ (n) $r = \frac{3}{2} - \sin\theta$ (o) $r = 1 - \sin\theta$ (p) $r = \frac{1}{2} - \sin\theta$

5. Sketch the roses:

   (a) $r = \sin 2\theta$ (b) $r = \sin 3\theta$ (c) $r = \sin 4\theta$ (d) $r = \sin 5\theta$
   (e) $r = \cos 2\theta$ (f) $r = \cos 3\theta$ (g) $r = \cos 4\theta$ (h) $r = \cos 5\theta$

6. Consider the equations $r = 2 + \sin\theta$ and $r = -2 + \sin\theta$.

   (a) Show that both the equations describe the same curve.

   (b) Sketch the curve.

7. Consider the equations $r = \sin\frac{\theta}{2}$ and $r = \cos\frac{\theta}{2}$.

   (a) Show that if (r, θ) satisfies the equation $r = \sin\frac{\theta}{2}$ then its one of the other representations $(-r, \theta + \pi)$ satisfies the equation $r = \cos\frac{\theta}{2}$.

   (b) Show that both the equations describe the same curve and sketch the curve.

   (c) Observe from the graph that the curve is symmetric with respect to both x-axis and y-axis.

8. Sketch the following curves:

   (a) $r = 2 + \sin(2\theta)$ (b) $r^2 = -\sin\theta$ (c) $r = \theta,\ \theta \geq 0$
   (d) $r = \theta, \theta \leq 0$ (e) $r = \theta$ (f) $r = -\theta$

9. Consider the equation $r = \theta + 2\pi$.

   (a) Observe that the equation changes if (r, θ) is replaced by $(r, \pi - \theta)$ or $(-r, -\theta)$.

   (b) Show that the equation given above and $r = \theta$ describe the same curve (Spiral of Archimedes).

   (c) Show that the curve obtained is symmetric with respect to the y-axis.

10. Sketch the regions described by the following sets.

    (a) $\{(r, \theta) : 1 \leq r \leq 1 - 2\cos\theta,\ \frac{\pi}{2} \leq \theta \leq 3\frac{\pi}{2}\}$
    (b) $\{(r, \theta) : 1 + \cos\theta \leq r \leq 3\cos\theta,\ \frac{-\pi}{3} \leq \theta \leq 0\}$

11. Replace the equation $x^2 + y^2 - 4y = 0$ by equivalent polar equation.

12. Replace the equation $r = 6\cos\theta + 8\sin\theta$ by equivalent Cartesian equation and show that the equation describe a circle.

Practice Problems 19 : Hints/Solutions

1. Solving $\sin x = \cos x$ implies that $x = \frac{\pi}{4} \in [0, \frac{\pi}{2}]$ (see Figure 1). Therefore the required area is $\int_0^{\frac{\pi}{4}} (\cos x - \sin x)dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x - \cos x)dx = 2\sqrt{2} - 2$.

2. (a) Note that $x^3 - 9x - 9 + x^2 = (x + 3)(x + 1)(x - 3)$.
   (b) Area $= \int_{-3}^{1} [(x^3 - 9x) - (9 - x^2)]dx + \int_{-1}^{3} [(9 - x^2) - (x^3 - 9x)]dx$ (see Figure 2).

3. See Figure 3.

4. See Figure 4 for the graphs of the equations given in (a)-(d),

   Observe that the graphs of the equations given in (e)-(h) are obtained by rotating the curves described by the equations given in (a)-(d) counterclockwise by π. For example, $r = 3 - \cos\theta = 3 + \cos(\theta - \pi)$.

   Similarly, the graphs of the equations given in (i)-(p) are obtained by rotating the curves described by the equations given in (a)-(h) counterclockwise by $\frac{\pi}{2}$. For example, $r = 3 + \sin\theta = 3 + \cos(\theta - \frac{\pi}{2})$.

5. See Figure 5.

6. (a) Observe that both the curves are symmetric with respect to the y-axis. Moreover, if (r, θ) satisfies the equation $r = 2 + \sin\theta$ then $(-r, -\theta)$ satisfies the equation $r = -2 + \sin\theta$ and vice versa. Therefore both the equations describe the same curve.
   (b) Refer Figure 6 for the graph.

7. (a) Easy to verify.
   (b) From (a), it follows that both the equations describe the same curve. Refer Figure 7 for the graph.
   (c) The symmetry is shown in the figure.

8. See Figure 8

9. It is easy to verify.

10. See Figure 10.

11. Substituting $x = r\cos\theta$ and $y = r\sin\theta$ in the given equation leads to the equation $r(r - 4\sin\theta) = 0$. The equation $r = 0$ represents the origin which is included in the curve described by the equation $r = 4\sin\theta$. The required equation is $r = 4\sin\theta$.

12. The given equation can be written as $r^2 - 6r\cos\theta - 8r\sin\theta = 0$. The substitutions, $x = r\cos\theta, y = r\sin\theta$ and $r^2 = x^2 + y^2$ lead to the equation $(x - 3)^2 + (y - 4)^2 = 25$.