Practice Problems 20 : Area in Polar coordinates, Volume of a solid by slicing

1. Consider the curves $r = \cos 2\theta$ and $r = \frac{1}{2}$.

   (a) Find the points of intersection of the curves.

   (b) Show that $(-\frac{1}{2}, \frac{\pi}{3}), (-\frac{1}{2}, \frac{2\pi}{3}), (-\frac{1}{2}, \frac{4\pi}{3})$ and $(-\frac{1}{2}, \frac{5\pi}{3})$ satisfy the equations $r = -\frac{1}{2}$ and $r = \cos 2\theta$.

2. Find the areas of the regions enclosed by the following curves.

   (a) $r = 1 + \cos\theta$

   (b) $r^2 = 9\cos 2\theta$

   (c) $r = 3\sin 3\theta$

3. In each of the following cases, find the area of the region that lies inside both the curves.

   (a) $r = 2,\ r = 4\sin\theta$

   (b) $r = 2\sin\theta,\ r = 2 - 2\sin\theta$

   (c) $r = 3,\ r = 6\cos 2\theta$.

4. In each of the following cases, find the area of the region that lies inside the first curve and outside the second curve.

   (a) $r = 2,\ r = 4\sin\theta$.

   (b) $r = 2\sin\theta,\ r = 2 - 2\sin\theta$

5. Find the areas of the regions described by the following sets.

   (a) $\{(r,\theta) : 0 \le r \le 2\sin\theta,\ 0 \le \theta \le \frac{\pi}{6}\} \bigcup \{(r,\theta) : 0 \le r \le 2 - 2\sin\theta,\ \frac{\pi}{6} \le \theta \le \frac{\pi}{2}\}$

   (b) $\{(r,\theta) : 0 \le r \le 2\sin 2\theta,\ 0 \le \theta \le \frac{\pi}{12}\} \bigcup \{(r,\theta) : 0 \le r \le 1,\ \frac{\pi}{12} \le \theta \le \frac{\pi}{4}\}$

6. Find the area of the region inside the outer loop and outside the inner loop of $r = 2+4\cos\theta$.

7. The base of a solid is the region bounded by $x = 0, y = 0, x = \frac{\pi}{2}$ and the curve $y = \sin x$. Each cross section of the solid perpendicular to the x-axis is an equilateral triangle with one side in the base of the solid. Find the volume of the solid.

8. A pyramid has a square base. Suppose the height of the pyramid is 4 meters and the side of the square base is 2 meters. Determine the volume of the pyramid by slicing method.

9. Consider the sphere of radius r centered at 0 and the two great circles of the sphere lying on the xy and xz planes. A part of the sphere is shaved off in a such a manner that the cross section of the remaining part, perpendicular to the x-axis, is a square with vertices on the great circles. Compute the volume of the remaining part.

10. Find the volume of the solid enclosed by the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$.

11. Find the volume of the solid enclosed by the ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$ and the planes $y = \sqrt{3}$ and $y = 1$.

Practice Problems 20 : Hints/Solutions

1. Solving $r = \cos 2\theta$ and $r = \frac{1}{2}$ gives $\theta = \frac{\pi}{6} + \pi k$ and $\theta = \frac{5\pi}{6} + \pi k,\ k \in \mathbb{Z}$. Therefore we get, $\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$ and $\frac{11\pi}{6}$. By symmetry, we see that the points of intersection occur at $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$ and $\frac{5\pi}{3}$ (see Figure 1). We can also see, by solving the equations $r = -\frac{1}{2}$ and $r = \cos 2\theta$, that the points of intersection occur at $\theta = \frac{\pi}{3} + \pi k$ and $\theta = \frac{2\pi}{3} + \pi k, k \in \mathbb{Z}$, that is $\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}$ and $\frac{5\pi}{3}$.

2. (a) The area is $\int_0^{2\pi} \frac{1}{2}(1+\cos\theta)^2 d\theta$. See Figure 2(a)

   (b) The area is $4\int_0^{\frac{\pi}{4}} \frac{1}{2}(9\cos 2\theta)d\theta$. See Figure 2(b)

   (c) The area is $3\int_0^{\frac{\pi}{3}} \frac{1}{2}(3\sin 3\theta)^2 d\theta$. See Figure 2(c)

3. (a) Solving $2 = 4\sin\theta$ and $\theta \in [0, \frac{\pi}{2}]$ implies that $\theta = \frac{\pi}{6}$. The required area is $2\left[\int_0^{\frac{\pi}{6}} \frac{1}{2}4^2\sin^2\theta d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}2^2 d\theta\right]$. See Figure 3(a).

   (b) Solving $2\sin\theta = 2 - 2\sin\theta$ and $\theta \in [0, \frac{\pi}{2}]$ implies that $\theta = \frac{\pi}{6}$. The required area is $2\left[\int_0^{\frac{\pi}{6}} \frac{1}{2}(2\sin\theta)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2-2\sin\theta)^2 d\theta\right]$. See Figure 3(b)

   (c) Solving $6\cos 2\theta = 3$ and $\theta \in [0, \frac{\pi}{2}]$ implies that $\theta = \frac{\pi}{6}$. The required area is $8\left[\int_0^{\frac{\pi}{6}} \frac{1}{2}3^2 d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{1}{2}(6\cos 2\theta)^2 d\theta\right]$. See Figure 3(c)

4. (a) Solving $2 = 4\sin\theta$ implies that $\theta = \frac{\pi}{6}$ and $\theta = \frac{5\pi}{6}$. The area is $\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \frac{1}{2}(16\sin^2\theta - 4)d\theta$. See Figure 4(a).

   (b) Solving $2\sin\theta = 2 - 2\sin\theta$ and $\theta \in [0, \frac{\pi}{2}]$ implies that $\theta = \frac{\pi}{6}$. The required area is $2\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{1}{2}\left[4\sin^2\theta - (2-2\sin\theta)^2\right] d\theta$. See Figure 4(b)

5. (a) Solving $2\sin\theta = 2 - 2\sin\theta$ and $\theta \in [0, \frac{\pi}{2}]$ implies that $\theta = \frac{\pi}{6}$. The required area is $\int_0^{\frac{\pi}{6}} \frac{1}{2}(2\sin\theta)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} (2-2\sin\theta)^2 d\theta$. See Figure 5(a)

   (b) Solving $2\sin 2\theta = 1$ and $\theta \in [0, \frac{\pi}{2}]$ implies that $\theta = \frac{\pi}{12}$. The required area is $\int_0^{\frac{\pi}{12}} \frac{1}{2}(2\sin 2\theta)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} d\theta$. See Figure 5(b)

6. Solving $r = 0$ and $r = 2 + 4\cos\theta$ gives that $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$. See Figure 6. The required area is $2\left[\int_0^{\frac{2\pi}{3}} \frac{1}{2}(2+4\cos\theta)^2 d\theta - \int_{\pi}^{\frac{4\pi}{3}} \frac{1}{2}(2+4\cos\theta)^2 d\theta\right]$ or
$$2\left[\int_0^{\frac{2\pi}{3}} \frac{1}{2}(2+4\cos\theta)^2 d\theta - \int_{\frac{2\pi}{3}}^{\pi} \frac{1}{2}(2+4\cos\theta)^2 d\theta\right].$$

7. For every $x \in [0, \frac{\pi}{2}]$, $A(x) = \frac{\sqrt{3}}{4}\sin^2 x$. The volume is $\int_0^{\frac{\pi}{2}} A(x)dx = \frac{\sqrt{3}}{16}\pi$. See Figure 7.

8. Let the pyramid be as in Figure 8. The area of the cross section (of the solid) by the plane $x = t$ is $A(t) = \frac{t^2}{4}$. The required volume is $\int_0^4 A(t)dt$.

9. The cross section of the solid by the plane $x = t$ is a square with side $\sqrt{2(r^2 - t^2)}$. Hence the area of the cross section $A(t) = 2(r^2 - t^2)$. The required volume is $\int_{-r}^{r} A(t)dt = \frac{8r^3}{3}$. See Figure 9.

10. See Figure 10. Observe that the solid lies between the planes $x = -1$ and $x = 1$. For any fixed $t \in [-1, 1]$ the cross section of the solid by the plane $x = t$ is a square given by $\{(t, y, z) : |y| \le \sqrt{1-t^2} \text{ and } |z| \le \sqrt{1-t^2}\}$. Therefore the area of the cross section $A(t) = 4(1-t^2)$. The required volume is $\int_{-1}^{1} A(t)dt = 4\int_{-1}^{1}(1-t^2)dt = \frac{16}{3}$.

11. For any $t \in [1, \sqrt{3}]$, the cross section of the solid by the plane $y = t$ is the ellipse $\frac{x^2}{9} + z^2 = 1 - \frac{t^2}{4}$ and its area $A(t)$ is $3\pi(1 - \frac{t^2}{4})$. The required volume is $\int_1^{\sqrt{3}} 3\pi(1 - \frac{t^2}{4})dt$.