PP 24 : Calculus of Vector Valued Functions I : Parametric representations of curves

1. Match the parametric equation with the curve it defines. The curves are given in Figures 1-11.
   (a) $R(t) = (t^2, t^3), t \in \mathbb{R}$ (Cuspidal cubic).
   (b) $R(t) = (e^t \cos t, e^t \sin t), t \geq 0$ (Logarithmic spiral)
   (c) $R(t) = (t \cos t, t \sin t), t \geq 0$ (Spiral)
   (d) $R(t) = (t^2 - 1, t(t^2 - 1)), t \in \mathbb{R}$ (Crunodal cubic)
   (e) $R(t) = (t^2 + t, 2t - 1), t \in \mathbb{R}$ (Parabola)
   (f) $R(t) = (\cos^3 t, \sin^3 t), 0 \leq t \leq 2\pi$ (Astroid)
   (g) $R(t) = (\sin^2 t, 2\cos t), t \in \mathbb{R}$
   (h) $R(t) = (\cos t^2, \sin t^2, t^2), t \in \mathbb{R}$
   (i) $R(t) = (\cos t, \sin t, \sin t), t \in \mathbb{R}$
   (j) $R(t) = (t \cos t, t \sin t, t), t \geq 0$
   (k) $R(t) = (1 + \sin t, 1 + \sin t, 2 + \sin t), t \in \mathbb{R}$

2. Find parametric representations of the following circles.
   (a) The circle of radius 4 centered at (1,0,2) and parallel to the yz -plane.
   (b) The circle of radius 3 centered at (0,0,0) and lying on the plane containing two unit vectors $\mathbf{u}$ and $\mathbf{v}$ where $\mathbf{u} \cdot \mathbf{v} = 0$.
   (c) The circle of radius 3 centered at (1,1,2) and parallel to the plane containing two unit vectors $\mathbf{u}$ and $\mathbf{v}$ where $\mathbf{u} \cdot \mathbf{v} = 0$.
   (d) The intersection of the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = y$.
   (e) The circle passing through $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$ and $e_3 = (0, 0, 1)$.

3. Parameterize the curve given by $x^3 + y^3 = 3xy$ by considering the parameter $t = \frac{y}{x}$ which is the slope of the line through the origin and the point (x, y) on the curve.

4. Consider the unit circle $x^2 + y^2 = 1$. By considering the parameter $t = \frac{y}{x-1}$ which is the slope of the line joining $(1, 0)$ and the point (x, y) on the curve, show that $R(t) = \left(\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1}\right)$ is a parametric representation of the unit circle. (This parametrization of the circle is called *rational parametrization*).

5. Consider a parametric representation of the line $R(t) = (x_0 + tu, y_0 + tv), t \in \mathbb{R}, (u, v) \neq (0, 0)$. Show that $vx - uy - vx_0 + uy_0 = 0$ is an implicit equation of the line.

6. Reparameterize the following curves in terms of arc length.
   (a) $R(t) = (2 + t, 3 - t, 5t), t \geq 0$.
   (b) $R(t) = (2\cos t, 2\sin t, \sqrt{5}t), t \geq 0$.

7. Find two parametric representations $R_1(t)$ and $R_2(t)$ for the line $y = x$ in $\mathbb{R}^2$ such that $R_1(0) = R_2(0) = (0, 0)$ and $R_1'(0) \neq (0, 0)$ but $R_2'(0) = (0, 0)$.

8. Consider a curve $R(t), t \in I$ and let $R'(t) \neq 0$ for all t. Show that the arc length parametrization $R(t(s))$ of the curve $R(t)$ has unit speed, i.e, $\|\frac{dR}{ds}\| = 1$.

Practice Problems 24 : Hints/Solutions

1. The curve is sketched/identified by plotting the points $R(t_i)$ for some $t_1, t_2, ..., t_n$.

   (a) Note that the curve is symmetric about the x axis i.e. if $(x(t), y(t))$ lies on the curve then $(x(t), -y(t)) = (x(-t), y(-t))$ also lies on the curve. Moreover $x(t) = t^2 > 0$ for all t. The curve is given in Figure 4.

   (b) Note that $R(t) = (r(t)\cos t, r(t)\sin t), t \geq 0$ where $r(t) = e^t$. So $R(t)$ is a parametric form of the polar curve $r(t) = e^t$. The curve is given in Figure 6.

   (c) The curve is given in Figure 5. It is a polar curve $r(t) = t, t \geq 0$.

   (d) For $t = 1$ and $t = -1$, $R(t) = (0, 0)$. The curve is symmetric about the x-axis. The curve is given in Figure 1.

   (e) Since $t = \frac{y+1}{2}$, we get $x = \frac{y^2}{4} + y + \frac{3}{4}$ (by eliminating t). The curve is given in Figure 3.

   (f) The curve is given in Figure 2.

   (g) Note that $4x + y^2 = 4, 0 \leq x \leq 1$ and $-2 \leq y \leq 2$. So the curve is a portion of a parabola which is given in Figure 7.

   (h) Observe that the x and y components trace out a circle in the xy-plane. The curve is given in Figure 9.

   (i) The x and y components trace out a circle and the curve lies on the plane $z = y$. The curve is given in Figure 8.

   (j) A point (x, y, z) on the curve satisfies the equation $x^2 + y^2 = z^2$. The curve is given in Figure 11.

   (k) If we substitute $t' = \sin t$, the points in the curve are represented by $(1+t', 1+t', 2+t')$ which lies on a straight line. Since $\sin t$ is bounded the given curve is a line segment which is given in Figure 10.

2. (a) The given circle is a translation of the circle $r(t) = (0, 4\cos t, 4\sin t)$. A parametrization of the given circle is $R(t) = (1, 0, 2) + (0, 4\cos t, 4\sin t), 0 \leq t \leq 2\pi$.

   (b) Observe that any point $\mathbf{p}$ on the plane containing $\mathbf{u}, \mathbf{v}$ and $(0, 0, 0)$ can be expressed as $\mathbf{p} = (\mathbf{p} \cdot \mathbf{u})\mathbf{u} + (\mathbf{p} \cdot \mathbf{v})\mathbf{v}$ (see PP 23). Let (x, y, z) be a point on the circle and t be the angle between the vectors (x, y, z) and $\mathbf{u}$. Then $(x, y, z) = 4(\cos t)\mathbf{u} + 4(\sin t)\mathbf{v}$. Therefore a parametric representation of the given circle is $R(t) = 4(\cos t)\mathbf{u} + 4(\sin t)\mathbf{v}$.

   (c) By (b), a parametrization of the given circle is $R(t) = (1, 1, 2) + 4(\cos t)\mathbf{u} + 4(\sin t)\mathbf{v}$.

   (d) Observe that the intersection is a circle lying in the plane $z = y$ centered at $(0, 0, 0)$ with radius 2. Let $\mathbf{u} = (1, 0, 0)$ and $\mathbf{v} = \frac{1}{\sqrt{2}}(0, 1, 1)$. Then $\mathbf{u}$ and $\mathbf{v}$ are perpendicular unit vectors lying on the plane $z = y$. Following the solution of (b), we observe that a parametric representation of the given circle is $R(t) = (2\cos t)(1, 0, 0) + (2\sin t)\frac{1}{\sqrt{2}}(0, 1, 1) = 2(\cos t, \frac{\sin t}{\sqrt{2}}, \frac{\sin t}{\sqrt{2}})$.

   (e) The center of the circle is the center of the equilateral triangle formed by e_1, e_2 and e_3 which is $\mathbf{u} = \frac{1}{3}(1, 1, 1)$. This can be easily checked because $\|u - e_1\| = \|u - e_2\| = \|u - e_3\| = \frac{\sqrt{6}}{3}$ and the point $\frac{1}{3}(1, 1, 1)$ lies on the triangular region. The unit vector in the direction from the center $\mathbf{u}$ towards the direction of a point on the circle e_3 is $\mathbf{w} = \frac{1}{\sqrt{6}}(1, 1, -2)$. If $\mathbf{v}$ is a unit vector which is perpendicular to $\mathbf{w}$ and $(1, 1, 1)$ which is a normal to the plane containing e_1, e_2, e_3, then $\mathbf{v} = \frac{1}{\sqrt{2}}(1, -1, 0)$. Following the solution of (c), we observe that a parametric representation of the given circle is $R(t) = \frac{1}{3}(1, 1, 1) + \frac{\sqrt{6}}{3}(\cos t)\mathbf{w} + \frac{\sqrt{6}}{3}(\sin t)\mathbf{v}$.

3. Substitute $y = tx$ into the equation and get $x = \frac{3t}{1+t^3}$, by ignoring the trivial solution $x = 0$. Since $y = tx$ we get $y = \frac{3t^2}{1+t^3}$. Therefore a parametrization for the curve is $R(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3}\right)$.

4. Substitute $y = t(x-1)$ into the equation and get $x = \frac{t^2-1}{t^2+1}$, by ignoring the trivial solution $x = 1$.

5. Let (x, y) be any point on the line. Then $(x - x_0, y - y_0) \times (u, v) = 0$.

6. (a) By definition $s(t) = \int_0^t \sqrt{(x'(\tau))^2 + (y'(\tau))^2 + (z'(\tau))^2} d\tau = \int_0^t \sqrt{27} d\tau = \sqrt{27}t$. This implies that $R(t(s)) = (2 + \frac{1}{\sqrt{27}}s, 3 - \frac{1}{\sqrt{27}}s, \frac{5}{\sqrt{27}}s)$.

   (b) By definition $s(t) = 3t$. Therefore $t(s) = \frac{s}{3}$ and hence $R(t(s)) = (2\cos\frac{s}{3}, 2\sin\frac{s}{3}, \sqrt{5}\frac{s}{3})$.

7. Consider $R_1(t) = (t, t)$ and $R_2(t) = (t^3, t^3)$, $t \in \mathbb{R}$.

8. Since the parametrization is in terms of s, $\|\frac{dR}{ds}\|$ is the speed of $R(t(s))$. We know that $\frac{dR}{ds} = T$ and therefore $\|\frac{dR}{ds}\| = 1$.