PP 25 : Calculus of Vector Valued Functions II : Tangent, normal and curvature

1. Consider the curve $R(t) = (t^2 - 1, t(t^2 - 1)), t \in \mathbb{R}$. Show that $R(-1) = R(1)$ and find the tangent lines for the curves at $R(1)$ and $R(-1)$.

2. Let $R(t) = (t^2 - 2t, t^2 + 2t)$. Find the points on the curve where the curve has either vertical or horizontal tangent.

3. Consider the curve $R_1(t) = (t, 1 - t, 3 + t^2)$ and $R_2(t) = (3 - t, t - 2, t^2)$
   (a) Find the points of intersections of the curves.
   (b) Find the angle between the curves at the points of intersection.

4. Suppose that a particle moves along the curve $R(t) = (e^t, e^{2t}, \sin t)$ from $t = 0$ to $t = 1$ and then it moves on the tangent line to the curve at $R(1)$ in the direction of the tangent vector. Find the position of the particle at $t = 5$.

5. Consider the curves $R_1(\theta) = ((\frac{3}{2} + \cos\theta)\cos\theta, (\frac{3}{2} + \cos\theta)\sin\theta)$ and $R_2(\theta) = ((3 + \cos\theta)\cos\theta, (3 + \cos\theta)\sin\theta), 0 \le \theta \le 2\pi$.
   (a) Represent the curves in polar forms.
   (b) Show that there exist two distinct elements $\theta_1, \theta_2 \in [\frac{\pi}{2}, \pi]$ such that the curve has vertical tangents at $R_1(\theta_1)$ and $R_1(\theta_2)$.
   (c) Show that there exists a unique $\theta \in [\frac{\pi}{2}, \pi]$ such that the curve has a vertical tangent at $R_2(\theta)$.
   (d) Sketch the curves.

6. Let T denote the unit tangent vector of the curve given by $R(t)$. Denote $R'(t), R''(t), T(t)$ and $T'(t)$ simply by R', R'', T and T'. Show that (under the assumptions that R'' and T exist).
   (a) $R''(t) = T'\frac{ds}{dt} + T\frac{d^2s}{dt^2}$
   (b) $R'' \times R' = \left(\frac{ds}{dt}\right)^2 T' \times T$
   (c) $\|T'\| = \frac{\|R'' \times R'\|}{\|R'\|^2}$
   (d) the curvature $\kappa = \frac{\|R'' \times R'\|}{\|R'\|^3}$.

7. For the following curves, find the unit tangent vector, principal normal and curvature.
   (a) $R(t) = (\sqrt{2}\cos t, \sin t, \sin t), t \in \mathbb{R}$
   (b) $R(t) = (\cos 2t, 2t, \sin 2t), t \in \mathbb{R}$
   (c) $R(t) = (t^2, \sin t - t\cos t, \cos t + t\sin t), t > 0$.

8. For each of the following curves, find a point on the curve at which the curvature is maximum.
   (a) $y = \ln x, x > 0$
   (b) $y = e^x, x \in \mathbb{R}$.
   (c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $0 < b < a$.

9. Let R(s) be an arc length parameter of a curve. Show that the curvature of the curve at a point $R(s)$ is given by $\|R''(s)\|$.

Practice Problems 25 : Hints/Solutions

1. It is clear that $R(-1) = R(1) = (0,0)$. Since $R'(t) = (2t, 3t^2 - 1), R'(1) = (2,2)$ and $R'(-1) = (-2,2)$ and hence $y = x$ and $y = -x$ are the tangent lines at $R(1)$ and $R(-1)$ respectively.

2. Since $\frac{dx}{dt} \neq 0$ and $\frac{dy}{dt} = 0$ at $t = -1$, the curve has a horizontal tangent at $R(-1) = (3,-1)$. Similarly, the curve has a vertical tangent at $R(1) = (-1,3)$.

3. Consider the second curve as $R_2(u)$ with parameter u. If $R_1(t) = R_2(u)$, then $t = 3-u, 1-t = u-2$ and $3+t^2 = u^2$. This implies that the curves meet at $R_1(1) = R_2(2) = (1,0,4)$. If θ is the angle between the tangent vector then $\cos\theta = \frac{R_1'(1)\cdot R_2'(2)}{\|R_1'(1)\|\|R_2'(2)\|} = \frac{1}{\sqrt{3}}$.

4. The tangent line at $R(1)$ is defined by $X(t) = (e, e^2, \sin 1) + t(e, 2e^2, \cos 1)$. Note that $X(0) = R(1)$. The position vector of the particle at $t = 5$ is $X(4)$.

5. (a) The polar forms of the curves R_1 and R_2 are $r_1(\theta) = \frac{3}{2} + \cos\theta$ and $r_2(\theta) = 3 + \cos\theta$.
   (b) If we consider $R_1(\theta) = (x_1(\theta), y_1(\theta))$ then in $[0,\pi]$, $\frac{dx_1}{d\theta} = 0$ at $\theta = \pi$ and $\theta = \cos^{-1}(\frac{-3}{4})$. Moreover $\frac{dy_1}{d\theta} \neq 0$ at these points.
   (c) If we consider $R_2(\theta) = (x_2(\theta), y_2(\theta))$ then in $[0,\pi]$, $\frac{dx_2}{d\theta} = 0$ only at $\theta = \pi$.
   (d) The curves are given in Practice Problems 19.

6. (a) This follows from the fact that $R'(t) = \frac{dR}{ds}\frac{ds}{dt} = T\frac{ds}{dt}$.
   (b) Use (a) and $T \times T = 0$.
   (c) Since T and T' are orthogonal, $\|T' \times T\| = \|T'\|\|T\| = \|T'\|$. Now use (b).
   (d) This follows from the definition of the curvature $\kappa = \frac{\|T'\|}{\|R'\|}$.

7. (a) $T(t) = \frac{R'(t)}{\|R'(t)\|} = \frac{1}{\sqrt{2}}\left(-\sqrt{2}\sin t, \cos t, \cos t\right)$, $N(t) = \frac{T'(t)}{\|T'(t)\|} = \frac{1}{1}\left(-\cos t, -\frac{2}{\sqrt{2}}\sin t, -\frac{2}{\sqrt{2}}\sin t\right)$ and $\kappa(t) = \frac{\|T'(t)\|}{\|R'(t)\|} = \frac{1}{\sqrt{2}}$.
   (b) $T(t) = \left(-\frac{\sin 2t}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\cos 2t}{\sqrt{2}}\right)$, $N(t) = (-\cos 2t, 0, -\sin 2t)$ and $\kappa(t) = \frac{1}{2}$.
   (c) $T(t) = \frac{1}{\sqrt{5}}(2, \sin t, \cos t)$, $N(t) = (0, \cos t, -\sin t)$ and $\kappa(t) = \frac{1}{5t}$

8. (a) Note that $\kappa(x) = \frac{|f''(x)|}{[1+(f'(x))^2]^{\frac{3}{2}}} = \frac{x}{(1+x^2)^{\frac{3}{2}}}$ and $\kappa'(x) = \frac{1-2x^2}{(1+x^2)^{\frac{5}{2}}}$. Verify that the curvature is maximum at $(\frac{1}{\sqrt{2}}, \ln\frac{1}{\sqrt{2}})$.
   (b) Observe that $\kappa(x) = \frac{e^x}{(1+e^{2x})^{\frac{3}{2}}}$ and $\kappa'(x) = \frac{e^x(1+e^{2x})^{\frac{1}{2}}(1-2e^{2x})}{(1+e^{2x})^3}$. Verify that the curvature is maximum at $(\frac{1}{2}\ln\frac{1}{2}, \frac{1}{\sqrt{2}})$.
   (c) Consider the ellipse as a parametric curve $R(t) = (a\cos t, b\sin t), 0 \leq t \leq 2\pi$. Using the formula for $\kappa(t) = \frac{\|R''(t)\times R'(t)\|}{\|R'(t)\|}$, obtain, $\kappa(t) = \frac{ab}{\left(\sqrt{a^2\sin^2 t + b^2\cos^2 t}\right)^3}$. Observe that $a^2\sin^2 t + b^2\cos^2 t \geq b^2$ for all $t \in [0,2\pi]$ and at $t = 0$ (resp., $t = \pi$), $a^2\sin^2 t + b^2\cos^2 t = b^2$. Therefore the maximum of $\kappa(t)$ is achieved at $t = 0$ and hence the curvature is maximum at $(a,0)$ (resp., $(-a,0)$).

9. Follows from the definition of κ, $\kappa = \|\frac{dT}{ds}\| = \|\frac{d}{ds}(\frac{dR}{ds})\|$.