PP 26 : Functions of several variables : Sequences, continuity and partial derivatives

1. Find the limit of the sequence $((\sin\frac{1}{n}, e^{-\frac{1}{n^2}}, \sin(\frac{\pi}{2}-\frac{1}{n}))$.

2. Find
   (a) $\lim_{(x,y)\to(0,0)} \frac{3x^2|y|}{x^2+y^2}$.
   (b) $\lim_{(x,y,z)\to(0,0,0)} \frac{1-\cos(x+y+z)}{(x+y+z)^2}$.
   (c) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$.

3. Show that the following limits do not exist.
   (a) $\lim_{(x,y,z)\to(0,0,0)} \frac{(x+y+z)^2}{x^2+y^2+z^2}$
   (b) $\lim_{(x,y)\to(0,0)} \frac{xy\cos y}{4x^2+y^2}$.

4. Show that the function $f:\mathbb{R}^2\to\mathbb{R}$ defined by $f(x,y)=\frac{\sin(xy)}{xy}$ for $xy\neq 0$ and 1 for $xy=0$ is continuous on $\mathbb{R}^2$.

5. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x,y)=\frac{x^3y}{2x^4+y^2}$ for $(x,y)\neq 0$ and 0 for $(x,y)=(0,0)$. Show that the function f is continuous at $(0,0)$.

6. Let $f(x,y)=e^{-\frac{1}{|x-y|}}$ when $x\neq y$. How must f be defined for $x=y$ so that f is continuous on $\mathbb{R}^2$?

7. Find a function $f:\mathbb{R}^2\to\mathbb{R}$ such that $f(0,0)=1, f(1,0)=0$ and $0\leq f(x,y)\leq 1$ for all $(x,y)\in\mathbb{R}^2$.

8. Let $f:\mathbb{R}^3\to\mathbb{R}$ be continuous and $X_0\in\mathbb{R}^3$. If $f(X_0)>0$ show that there exists an $\epsilon-$neighborhood $B_\epsilon(X_0)=\{X\in\mathbb{R}^3:\|X-X_0\|<\epsilon\}$ of X_0 such that $f(X)>0$ for all $X\in B_\epsilon(X_0)$.

9. Let $f:\mathbb{R}^2\to\mathbb{R}$ be given by $f(x,y)=1$ if $x=0$ or $y=0$ and $f(x,y)=0$ otherwise. Show that $f_x(0,0)=f_y(0,0)=0$ but f is not continuous at $(0,0)$.

10. Let $f,g:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x,y)=|x|+|y|$ and $g(x,y)=|xy|$ for $(x,y)\in\mathbb{R}^2$. Show that
    (a) $f_x(0,0)$ and $f_y(0,0)$ do not exist whereas $g_x(0,0)$ and $g_y(0,0)$ exist.
    (b) for $x_0\neq 0$, $g_y(x_0,0)$ does not exist and for $y_0\neq 0$, $g_x(0,y_0)$ does not exist.

11. Consider the function $f(x,y)=\frac{3x^2y-y^3}{x^2+y^2}$ for $(x,y)\neq(0,0)$ and $f(0,0)=0$.
    (a) Verify whether f is continuous at $(0,0)$.
    (b) Evaluate $f_y(x,0)$ for $x\neq 0$.
    (c) Verify whether f_y is continuous at $(0,0)$.

12. Let $f:\mathbb{R}^2\to\mathbb{R}$ be such that $f(X+Y)=f(X)+f(Y)$ and $f(\alpha X)=\alpha f(X)$ for all $X,Y\in\mathbb{R}^2$ and $\alpha\in\mathbb{R}$. Show that f is continuous on $\mathbb{R}^2$.

13. (*) Let A be a bounded subset of $\mathbb{R}^2$. Suppose $(x_0,y_0)\in A$ whenever a sequence $((x_n,y_n))$ in A converges to (x_0,y_0). Let $f:A\to\mathbb{R}$ be continuous. Show that

(a) f is bounded.

(b) there exists $X_0, Y_0 \in A$ such that $f(X_0) = \sup\{f(X) : X \in A\}$ and $f(Y_0) = \inf\{f(X) : X \in A\}$.

14. (*) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous. Let $S = \{X \in \mathbb{R}^2 : \|X\| \leq 1\}$. Show that the range of f, $f(S) = \{f(X) : X \in S\}$, is an interval.

Practice Problems 26 : Hints/Solutions

1. $(\sin\frac{1}{n}, e^{-\frac{1}{n^2}}, \sin(\frac{\pi}{2} - \frac{1}{n})) \to (0, 0, 1)$ an $n \to \infty$.

2. (a) Since $\lim_{(x,0)\to(0,0)} \frac{3x^2|y|}{x^2+y^2} = 0$, 0 is the possible limit. Now $\left|\frac{3x^2|y|}{x^2+y^2} - 0\right| \leq \frac{3(x^2+y^2)|y|}{x^2+y^2} = |y| \to 0$ as $(x, y) \to (0, 0)$. Therefore 0 is the limit.

   (b) As $(x, y, z) \to (0, 0, 0,), t = x + y + z \to 0$. Therefore $\lim_{(x,y,z)\to(0,0,0)} \frac{1-\cos(x+y+z)}{(x+y+z)^2} = \lim_{t\to 0} \frac{1-\cos t}{t} = \frac{1}{2}$.

   (c) $\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2} = 1$.

3. (a) Along $x = y = 0$, $\lim_{(x,y,z)\to(0,0,0)} \frac{(x+y+z)^2}{x^2+y^2+z^2} = 1$ whereas, along $x = y = z$, $\lim_{(x,y,z)\to(0,0,0)} \frac{(x+y+z)^2}{x^2+y^2+z^2} = 3$. Therefore the limit does not exist.

   (b) For $x = 0$, $\lim_{(x,y)\to(0,0)} \frac{xy\cos y}{4x^2+y^2} = 0$ and for $x = y$, $\lim_{(x,y)\to(0,0)} \frac{xy\cos y}{4x^2+y^2} = \frac{1}{5}$. Therefore the limit does not exist.

4. Let $(x_0, y_0) \in \mathbb{R}^2$ and $x_0y_0 \neq 0$. The function f is continuous at (x_0, y_0) as $f(x_n, y_n) \to f(x_0, y_0)$ when $(x_n, y_n) \to (x_0, y_0)$. Suppose $(x_0, y_0) \in \mathbb{R}^2$ such that $x_0y_0 = 0$ and $(x_n, y_n) \to (x_0, y_0)$. Since $x_ny_n \to 0, f(x_n, y_n) \to 1 = f(x_0, y_0)$. Therefore f is continuous at (x_0, y_0).

5. By AM-GM inequality, $|f(x, y) - f(0, 0)| \leq \left|\frac{x^3y}{x^4+y^2}\right| \leq \frac{2x(x^4+y^2)}{x^4+y^2} \to 0$ as $(x, y) \to (0, 0)$.

6. Setting $f(x, y) = 0$ for $x = y$ makes the function continuous.

7. Consider $f(x, y) = \frac{|x+y-1|}{|x+y|+1}$.

8. Suppose that there exists no such ϵ-neighborhood. Then for every n, there exists $X_n \in B_{\frac{1}{n}}(X_0) = \{X \in \mathbb{R}^3 : \|X - X_0\| \leq \frac{1}{n}\}$ such that $f(X_n) \leq 0$. Since $X_n \to X_0$, by the continuity of $f, f(X_n) \to f(X_0)$. Therefore $f(X_0) \leq 0$ which is a contradiction.

9. Easily follows from the definitions.

10. For $t \neq 0$, $\frac{f(0+t,0)-f(0,0)}{t} = \frac{|t|}{t}$ and $\frac{f(0,0+t)-f(0,0)}{t} = \frac{|t|}{t}$. Therefore $f_x(0, 0)$ and $f_y(0, 0)$ do not exist. For $(x_0, y_0) \in \mathbb{R}^2$ and $t \neq 0$, $\frac{g(x_0+t,y_0)-g(x_0,y_0)}{t} = \frac{|y_0|(|x_0+t|-|x_0|)}{t}$. By allowing $t \to 0$, we see that $f_x(0, 0) = 0, f_y(0, 0) = 0$ and $f_x(0, y_0)$ does not exist if $y_0 \neq 0$.

11. (a) Since $|f(x, y) - f(0, 0)| \leq \frac{|y||3x^2-y^2|}{x^2+y^2} \leq \frac{|y||3x^2+3y^2|}{x^2+y^2} \leq 3|y| \to 0$ as $(x, y) \to (0, 0)$, f is continuous at $(0, 0)$.

    (b) $f_y(0, 0) = \lim_{t\to 0} \frac{f(0,t)}{t} = -1$.

    (c) Since $f_y(x, 0) = \lim_{t\to 0} \frac{f(x,t)-f(x,0)}{t} = 3$ for any $x \neq 0$, $f_y(x, 0) \not\to f_y(0, 0)$ as $x \to 0$. Therefore f_y is not continuous at $(x_0, y_))$.

12. Let $(x_0, y_0) \in \mathbb{R}^2$ and $(x_n, y_n) \to (x_0, y_0)$. Then $f((x_n, y_n)) = f(x_n(1,0) + y_n(0,1)) = x_n f(1,0) + y_n f(0,1) \to x_0 f((1,0)) + y_0 f((0,1)) = f((x_0, y_0))$.

13. (a) If f is not bounded then for every n, there exists $(x_n, y_n) \in A$ such that $f((x_n, y_n)) > n$. Since $((x_n, y_n))$ is a bounded sequence, there exists a subsequence $((x_{n_k}, y_{n_k}))$ such that $(x_{n_k}, y_{n_k}) \to (x_0, y_0) \in A$. By the continuity of f, $f((x_{n_k}, y_{n_k})) \to f(x_0, y_0)$ which contradicts the assumption that $f(x_n, y_n) > n$ for every n.

    (b) For every n, find $(x_n, y_n) \in A$ such that $\sup\{f(X) : X \in A\} - \frac{1}{n} \leq f(x_n, y_n)$. Since $((x_n, y_n))$ is bounded, by Bolzano-Weierstrass theorem, there exists a subsequence $(x_{n_k}, y_{n_k}))$ converges, say to $X_0 = (x_0, y_0)$. By the continuity of f, $f(x_n, y_n) \to f(x_0, y_0) \geq \sup\{f(X) : X \in A\}$, that is $f(X_0) = \sup\{f(X) : X \in A\}$.

14. Let $X_0, Y_0 \in S$ be such that $f(X_0) = M = \sup\{f(X) : X \in S\}$ and $f(Y_0) = m = \inf\{f(X) : X \in S\}$ (see Problem 13). Note that $f(X) \in [m, M]$ for every $X \in S$. Suppose $\alpha \in (m, M)$. Consider the map $g(t) = f((1-t)Y_0 + tX_0)$. Observe that $g : [0,1] \to \mathbb{R}$ is continuous, $g(0) = m$ and $g(1) = M$. By the intermediate value property, there exists $t_0 \in (0,1)$ such that $g(t_0) = \alpha$, that is $f((1-t_0)Y_0 + t_0 X_0) = \alpha$. Hence $f(S) = [m, M]$.