PP 27 : Functions of several variables : Differentiabilty and Chain Rule

1. In each of the following cases discuss the differentiabilty of f at $(0,0)$ where $f(x,y)$, for $(x,y) \in \mathbb{R}^2$, is

   (a) $|x| + |y|$

   (b) $\|(x,y)\|$

   (c) 0 if $xy \neq 0$ and 1 if $xy = 0$

   (d) $\sqrt{|xy|}$

   (e) $\frac{x^2y}{\sqrt{x^2+y^2}}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

   (f) $\frac{2x^2y}{x^2+y^2}$ if $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

   (g) $x^2 + \sin y + y^2e^x$

2. Let $f(x,y) = |xy|$ for all $(x,y) \in \mathbb{R}^2$. Show that

   (a) f is differentiable at $(0,0.)$

   (b) $f_x(0,y_0)$ does not exist if $y_0 \neq 0$.

3. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = (x-y)^2 \sin\frac{1}{x-y}$ if $x \neq y$ and $f(x,x) = 0$. Show that

   (a) f_x and f_y exist at all points of $\mathbb{R}^2$.

   (b) f is differentiable at $(0,0)$.

   (c) f_x and f_y are not continuous on the line $y = x$.

4. Let $z = f(x,y), x = r\cos\theta$ and $y = r\sin\theta$.

   (a) Show that $\frac{\partial z}{\partial r} = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$ and $\frac{1}{r}\frac{\partial z}{\partial \theta} = -\frac{\partial f}{\partial x}\sin\theta + \frac{\partial f}{\partial y}\cos\theta$.

   (b) If $f(x,y) = x^2 + 2xy$, show that $\frac{\partial z}{\partial \theta} = 2(x^2 - xy - y^2)$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable.

   (a) For $(x_0,y_0), (h,k) \in \mathbb{R}^2$, define $\phi : [0,1] \to \mathbb{R}$ by $\phi(t) = f(x_0 + th, y_0 + tk), t \in [0,1]$. Show that ϕ is differentiable and $\phi' = hf_x + kf_y$.

   (b) Suppose $f(1,2) = f(3,4) = 0$. Show that there exists a point (x_0,y_0) lying in the line segment joining $(1,1)$ and $(3,4)$ such that $f_x(x_0,y_0) = -f_y(x_0,y_0)$.

6. An ice block of rectangular shap is melting. Suppose that at a given instant, the block has a height of 5 ft, a length of 10 ft and a width of 12 ft. If each of the dimension is decreasing at a rate of 5 ft per hour, at what rate is the volume of the block decreasing at the given instant ?

7. (*) Let $f : \mathbb{R}^2 \to \mathbb{R}$ and $(x_0,y_0) \in \mathbb{R}^2$. Suppose that f_x and f_y exist in a neighborhood $N_\epsilon(x_0,y_0) = \{(x,y) : \|(x,y) - (x_0,y_0)\| < \epsilon\}$ of (x_0,y_0) for some $\epsilon > 0$. Suppose that f_x, f_y exist on $N_\epsilon(x_0,y_0)$ and f_x is continuous at (x_0,y_0).

   (a) Define $f_1(x,y) = \frac{f(x,y)-f(x_0,y)}{x-x_0}$ if $x \neq x_0$ and $f_1(x_0,y) = f_x(x_0,y)$. Define $f_2(x_0,y) = \frac{f(x_0,y)-f(x_0,y_0)}{y-y_0}$ if $y \neq y_0$ and $f_2(x_0,y_0) = f_y(x_0,y_0)$. Show that $f_1 : N_\epsilon(x_0,y_0) \to \mathbb{R}$ and $f_2(x_0,\cdot) : (y_0-\epsilon, y_0+\epsilon) \to \mathbb{R}$ are continuous at (x_0,y_0) and y_0 respectively.

(b) Show that $f(x,y) - f(x_0,y_0) = (x-x_0)f_1(x,y) + (y-y_0)f_2(x,y)$ for all $x,y \in N_\epsilon(x_0,y_0)$.

(c) Show that f is dfferentiable at (x_0,y_0).

Practice Problems 27 : Hints/Solutions

1. (a) Since $f_x(0,0)$ does not exist, f it is not differentiable at $(0,0)$.

   (b) The limit $\lim_{t\to 0} \frac{f(t,0)-f(0,0)}{t} = \lim_{t\to 0} \frac{|t|}{t}$ does not exist. Hence $f_x(0,0)$ does not exist and therefore f is not differentiable at $(0,0)$.

   (c) The function is not continuous at $(0,0)$ and hence it is not differentiable at $(0,0)$.

   (d) Observe that $f_x(0,0) = f_y(0,0) = 0$ and $\epsilon(h,k) = \frac{f(h,k)-f(0,0)-f_x(0,0)\cdot h - f_y(0,0)\cdot k}{\sqrt{h^2+k^2}} = \frac{\sqrt{|hk|}}{\sqrt{h^2+k^2}} \nrightarrow 0$ as $(h,k)\to(0,0)$ along the line $h=k$. Hence f is not differentiable at $(0,0)$.

   (e) Since $f_x(0,0) = 0$ and $f_y(0,0) = 0$, $|\epsilon(h,k)| = \left|\frac{h^2k}{\sqrt{h^2+k^2}}\right| \le |k| \to 0$ as $(h,k)\to(0,0)$. Therefore f is differentiable at $(0,0)$.

   (f) Note that $f_x(0,0) = f_y(0,0) = 0$ and $\epsilon(h,k) \to \frac{1}{\sqrt{2}}$ along $h=k$. Therefore f is not differentiable at $(0,0)$.

   (g) The partial derivatives of f are continuous. Hence f is differentiable.

2. (a) Observe that $f_x(0,0) = f_y(0,0) = 0$. Now $\epsilon(h,k) = \frac{f(h,k)-f(0,0)-f_x(0,0)\cdot h - f_y(0,0)\cdot k}{\sqrt{h^2+k^2}} = \frac{|hk|}{\sqrt{h^2+k^2}} \le \frac{\sqrt{h^2+k^2}|k|}{\sqrt{h^2+k^2}} \to 0$ as $(h,k)\to(0,0)$. Therefore f is differentiable at $(0,0)$.

   (b) For $y_0 \ne 0$, $\lim_{t\to 0} \frac{f(t,y_0)-f(0,y_0)}{t} = \lim_{t\to 0} \frac{|t||y_0|}{t}$ does not exist.

3. (a) For $x\ne y$, $f_x(x,y) = 2(x-y)\sin\frac{1}{x-y} - \cos\frac{1}{x-y}$ and $f_x(x,x) = 0$. Similarly, for $x \ne y$, $f_y(x,y) = -2(x-y)\sin\frac{1}{x-y} + \cos\frac{1}{x-y}$ and $f_y(x,x) = 0$.

   (b) Now $|\epsilon(h,k)| \le \left|\frac{(h-k)^2}{\sqrt{h^2+k^2}}\sin\frac{1}{h-k}\right| \le \frac{3(h^2+k^2)}{\sqrt{h^2+k^2}} \to 0$ as $(h,k)\to(0,0)$.

   (c) It is clear that f_x and f_y are not continuous at $(0,0)$.

4. (a) Use the chain rule, for example, $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$.

   (b) Use (a).

5. (a) Let $x(t) = x_0 + th$ and $y(t) = y_0 + tk$ and apply the chain rule $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$.

   (b) Take $(x_0,y_0) = (1,2)$ and $(h,k) = (2,2)$. Apply (a) and Rolle's Theorem for ϕ.

6. If $x(t), y(t)$ and $z(t)$ are height, length and width of the block at time t, then the volume is $V(t) = x(t)y(t)z(t)$. By the chain rule $\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} + \frac{\partial V}{\partial z}\frac{dz}{dt}$. Note that $\frac{dx}{dt} = \frac{dy}{dt} = \frac{dz}{dt} = -5, x(t_0) = 5, y(t_0) = 10$ and $z(t_0) = 12$ at the given time t_0. Therefore at t_0, the rate of change of the volume of the block is $\frac{dV}{dt}|t_0 = -460$ ft per hour.

7. (a) Note that $\lim_{y\to y_0} f_2(x_0,y) = f_y(x_0,y_0) = f_2(x_0,y_0)$. Therefore $f_2(x_0,\cdot)$ is continuous at y_0.

   For any $(x,y) \in N_\epsilon(x_0,y_0)$, $x \ne x_0$, by the MVT (of one variable) there exists c between x and x_0 such that $f(x,y) - f(x_0,y) = (x-x_0)f_x(c,y)$, that is $f_1(x,y) = f_x(c,y)$. Since f_x is continuous at (x_0,y_0), f_1 is continuous at (x_0,y_0).

(b) This is obvious.

(c) For $(h,k) \in \mathbb{R}^2$, consider $|\epsilon(h,k)| = \left|\frac{f(x_0+h,y_0+k)-f(x_0,y_0)-f_x(x_0,y_0)\cdot h-f_y(x_0,y_0)\cdot k}{\sqrt{h^2+k^2}}\right|$

By (b), we get that $|\epsilon(h,k)| \leq \left|\frac{|h||f_1(x_0+h,y_0+k)-f_x(x_0,y_0)|+|k||f_2(x_0+h,y_0+k)-f_y(x_0,y_0)|}{\sqrt{h^2+k^2}}\right|$

$\leq |f_1(x_0+h,y_0+k)-f_x(x_0,y_0)|+|f_2(x_0+h,y_0+k)-f_y(x_0,y_0)| \to 0$ as $(h,k) \to (0,0)$.

Therefore f is differentiable at (x_0,y_0).