PP 30 : Maxima, Minima, Second Derivative Test

1. Let $D \subset \mathbb{R}^2$ and (x_0, y_0) be an interior point of D. Suppose that $f : D \to \mathbb{R}$ and f has a local maximum or minimum at (x_0, y_0).

   (a) If $(u, v) \in \mathbb{R}^2$, $\|(u, v)\| = 1$ and $D_{(x_0,y_0)}f(u, v)$ exists, show that $D_{(x_0,y_0)}f(u, v) = 0$.

   (b) If f is differentiable at (x_0, y_0), show that $f'(x_0, y_0) = 0$.

2. Let $f(x, y) = 5y^4 - 6xy^2 + x^2$ for all $(x, y) \in \mathbb{R}^2$. Show that

   (a) f has a local minimum at (0,0) along every line through $(0, 0)$.

   (b) $D_{(0,0)}f(u, v) = 0$ for every $(u, v) \in \mathbb{R}^2$ satisfying $\|(u, v)\| = 1$.

   (c) $f'(0, 0) = 0$.

   (d) f does not have a local minimum at $(0, 0)$.

3. Examine the following functions for local maxima, local minima and saddle points.

   (a) $x^2 - y^2$

   (b) $x^4 + y^4 - 2x^2 - 2y^2 + 4xy$

   (c) $x^2 - 2xy^2$

4. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = xye^{-(x^2+y^2)}$ for all $(x, y) \in \mathbb{R}^2$.

   (a) Identify the points of local maxima and minima, and the saddle points.

   (b) Show that f is bounded on $\mathbb{R}^2$.

   (c) Show that the points of local maxima/minima are the points of absolute maxima/minima.

5. Show that $\int_0^1 (\sqrt{x} - \frac{4}{15} - \frac{4}{5}x)^2 dx = \inf\{\int_0^1 (\sqrt{x} - a - bx)^2 dx : a, b \in \mathbb{R}\}$ (The linear function y $= \frac{4}{15} + \frac{4}{5}x$ is called a "least square approximation" to $y = \sqrt{x}$ in the interval $[0, 1]$).

6. Find a point on the surface $z = xy + 1$ which is nearest to $(0, 0, 0)$.

7. Let $D = \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 0\}$ and $f : D \to \mathbb{R}$ be given by $f(x, y) = xy + \frac{1000}{x} + \frac{1000}{y}$. Find the infimum of the function $f(x, y)$ on D.

8. If we want to make a rectangular box, open at the top, with volume 500 cubic cms using least amount of material, what should be the dimensions of the box ?

9. Let $D = \{(x, y) \in \mathbb{R}^2 : x \geq 0 \text{ and } y \geq 0\}$ and $f : D \to \mathbb{R}$ be given by $f(x, y) = (x^2 + y^2)e^{-(x+y)}$. Show that

   (a) f is bounded on D.

   (b) f achieves its (absolute) maximum at a point on the boundary of D.

   (c) $e^{x+y-2} \geq \frac{x^2+y^2}{4}$ for all $(x, y) \in D$.

10. Find the points of absolute maximum and absolute minimum of the function $f(x, y) = x^2 + y^2 - 2x + 2$ on the region $\{(x, y) : x^2 + y^2 \leq 4 \text{ with } y \geq 0\}$.

11. Let $D = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0 \text{ and } x + y + z = 100\}$ and $f : D \to \mathbb{R}$ be defined by $f(x, y, z) = xyz$. Find the absolute maximum value of f on D.

<u>Practice Problems 30: Hints/Solutions</u>

1. (a) By Problem 5 of PP 28, $D_{(x_0,y_0)}f(u,v)$ is the derivative of $f(x_0+tu, y_0+tv)$ with respect to t at 0. Since the function $f(x_0+tu, y_0+tv)$ has a minimum at $t=0$, $D_{(x_0,y_0)}f(u,v)=0$.
   (b) Since $f_x(x_0,y_0)=f_y(x_0,y_0)=0$, $f'(x_0,y_0)=0$.

2. (a) For a fixed $m \in \mathbb{R}$, consider the line $y=mx$. Then on the line, $f(x,y)=f(x,mx)$ which is a function of one variable. Verify that the function $f(x,mx)$ has a local minimum at $x=0$.
   (b) Follows from Problem 1.
   (c) Follows from Problem 1.
   (d) For $\epsilon>0$, $f(0,\epsilon)=5\epsilon^4>0$ and $f(2\epsilon^2,\epsilon)<0$.

3. (a) Let $f(x,y)=x^2-y^2$. Note that $f_x(x,y)=f_y(x,y)=0$ if and only if $(x,y)=(0,0)$. Therefore $(0,0)$ is the only critical point for f. Since $(f_{xx}f_{yy}-f_{xy}^2)(0,0)<0$, the point $(0,0)$ is a saddle point. The function f has neither a point of local maximum nor a point of local minimum.
   (b) Let $f(x,y)=x^4+y^4-2x^2-2y^2+4xy$. By solving $f_x(x,y)=f_y(x,y)=0$, we get the critical points $(0,0), (-\sqrt{2},\sqrt{2})$ and $(\sqrt{2},-\sqrt{2})$. By the second derivative test both $(-\sqrt{2},\sqrt{2})$ and $(\sqrt{2},-\sqrt{2})$ are relative minima and the test fails for the critical point $(0,0)$. Along $y=0$, $f(x,y)=x^4-2x^2$ and therefore for sufficiently small $x\neq 0$, $f(x,0)<0$. Along $y=x$, $f(x,y)=2x^4$ and hence $f(x,x)>0$ for $x\neq 0$. Therefore $(0,0)$ is a saddle point.
   (c) Let $f(x,y)=x^2-2xy^2$. Observe that $(0,0)$ is the only critical point and it is a saddle point of f. Because, for $\epsilon>0$, $f(\epsilon,0)>0$ and $f(\epsilon^2,\epsilon)<0$.

4. (a) Solving $f_x=f_y=0$ implies that $y(1-2x^2)=0$ and $x(1-2y^2)=0$. So we get the critical points: $(0,0), (-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$, $(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$. Using the second derivative test we identify that $(0,0)$ is a saddle point; $(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$ and $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ are the points of local maxima and $(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ are the points of local minima.
   (b) Observe that $|f(x,y)|\leq \|(x,y)\|e^{-\|(x,y)\|^2}\to 0$ as $\|(x,y)\|\to\infty$. This shows that there exists $R>0$ such that $f(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})=-\frac{1}{2e}<f(x,y)<\frac{1}{2e}=f(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ for all (x,y) such that $\|(x,y)\|\geq R$. Since f is a continuous function, it is bounded on the disc $\{(x,y)\in\mathbb{R}^2 : \|(x,y)\|\leq R\}$. Therefore f is bounded on $\mathbb{R}^2$.
   (c) Since $f(x,y)$ is a continuous function, it attains its supremum and infimum on the closed and bounded disc $\{(x,y)\in\mathbb{R}^2 : \|(x,y)\|\leq R\}$. Therefore $(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ is a point of minimum and $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$ is a point of maximum of f.

5. Let $f(a,b)=\int_0^1(\sqrt{x}-a-bx)^2dx=a^2-\frac{4a}{3}+ab-\frac{4b}{5}+\frac{b^2}{3}+\frac{1}{2}$. By solving $f_a=f_b=0$, we get $a=\frac{4}{15}$ and $b=\frac{4}{5}$. We conclude from the second derivative test that $(\frac{4}{15},\frac{4}{5})$ is the point of minimum for f.

6. The given problem is to minimize the function $x^2+y^2+z^2$ subject to $z=xy+1$. If we consider x and y are independent variables and $z=xy+1$, the problem is reduced to minimizing the function $f(x,y)=x^2+y^2+(xy+1)^2$. By the first and second derivative tests, $(0,0)$ is a point of local minimum of f. The corresponding point on the surface is $(0,0,1)$. Since the nearest point to $(0,0,0)$ from the surface exists, $(0,0,1)$ has to be the nearest point.

7. Solving $f_x = f_y = 0$ on D implies that $(10, 10)$ is the only critical point in D. By the second derivative test, $(10, 10)$ is a point of local minimum of f on D. If we can justify that this is a point of (absolute) minimum of f on D, then $f(10, 10) = 300$ is the infimum of f. For justification, consider the subset R of D given by $R = \{(x, y) : 1 \leq x \leq 400, 1 \leq y \leq 400\}$. Observe that if $(x, y) \in D \setminus R$, then $f(x, y) > 300$. Since the minimum of the continuous function f on the closed bounded set R is achieved, $(10, 10)$ is the (absolute) minimum of f on R. From the above observation it follows that $(10, 10)$ is the absolute minimum of f on D.

8. If we let x, y and z be the length, width and height of the box respectively, then we want to minimize $xy+2xz+2yz$ subject to the constraint $xyz = 500$. Since $xy > 0$ and $z = \frac{500}{xy}$, we minimize the function $f(x, y) = xy+\frac{1000}{y}+\frac{1000}{x}$ over the set $\{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 0\}$. The rest follows from Problem 7. The required length, width and height of the box are 10, 10 and 5 cms respectively.

9. The solution to this problem is similar to the solution to Problem 4.
   (a) Since for $x, y > 0$, $(x^2 + y^2)e^{-(x+y)} \leq (x + y)^2 e^{-(x+y)} \to 0$ as $\|(x, y)\| \to \infty$ the function is bounded.
   (b) Solving $f_x = f_y = 0$ on D implies that $(1, 1)$ is the only critical point in the interior of D. On the boundary $\{(x, 0) : x > 0\}$, the function is x^2e^{-x} which attains its local maximum at $x = 2$. Similarly on the boundary $\{(0, y) : y > 0\}$, the function is y^2e^{-y} which attains its local maximum at $y = 2$. From the proof of (a) and comparing the values of $f(1, 1), f(0, 0)$, $f(2, 0)$ and $f(0, 2)$, we see that $(0, 2)$ and $(2, 0)$ are the points of maxima for f on D.
   (c) By (a), $4e^{-2} \geq (x^2 + y^2)e^{-(x+y)}$.

10. By solving $f_x = 0$ and $f_y = 0$, we see that there is no critical point in the interior of the region. On the curve $x^2 + y^2 = 4, y \geq 0$, the function is $x^2 + 4 - x^2 - 2x + 2 = -2x + 6$ where $x \in [-2, 2]$. For this function there is no critical point in the interval $(-2, 2)$ and therefore the candidates for the points of maxima/minima for f on the curve are $(-2, 0)$ and $(2, 0)$. On the line segment joining $(-2, 0)$ and $(2, 0)$, the function is $x^2 - 2x + 2$ where $x \in [-2, 2]$. The critical point for this function in the interior of $[-2, 2]$ is $x = 1$ and therefore the point $(1, 0)$ is also a candidate. Since $f(-2, 0) = 10$, $f(2, 0) = 2$ and $f(1, 0) = 1$, the point of maximum is $(-2, 0)$ and the point of minimum is $(1, 0)$.

    This problem can alternately be solved as follows. Note that the points of minima and maxima for $f(x, y)$ and the function $g(x, y) = (x - 1)^2 + y^2$ are same. Since the value $(x - 1)^2 + y^2$ is the distance between (x, y) and the point $(1, 0)$, the point of maximum is $(-2, 0)$ and the point of minimum is $(1, 0)$.

11. First note that D is a bounded subset of the pane $x+y+z = 100$. Since f is a continuous function on the bounded set D, a point of absolute maximum for f on D exists. Moreover, since the value of f on the boundary of D is zero, f attains its maximum in the interior of D. In the interior of D, $f(x, y, z) = xyz$ and $z = 100 - x - y$. So we maximize the function $g(x, y) = xy(100 - x - y)$ on $\{(x, y) : x > 0 \text{ and } y > 0\}$. From the first and second derivative tests we get the equations $x + 2y = 100$ and $y + 2x = 100$. This implies that g attains its local maximum at $(\frac{100}{3}, \frac{100}{3})$. Therefore, $(\frac{100}{3}, \frac{100}{3}, \frac{100}{3})$ is a point of local maximum for f in the interior of D. Since f attains its absolute maximum in the interior, $(\frac{100}{3}, \frac{100}{3}, \frac{100}{3})$ is the point of absolute maximum on D.

Let I be in interval in Rhaving more than one point, and let $f : I \to \mathcal{R}$ be differentiable. We know that f is increasing on I if and only if $f' \geq 0$ on I, and f is convex on I if and only if f' is increasing on I. Similarly, it will be nice to identify a 'geometric' property P of the function f so that f satisfies P on I if and only if f' is convex on I.

Let $I = (0, \infty)$, $a \in \mathcal{R}$ and $f(x) = x^a$ for $x \in I$. One may observe that f' is convex on I if and only if $0 \leq a \leq 1$ or $a \geq 2$. This example shows that the property P has to be rather subtle!