# PP 32 : Double integral

1. Let $R = [a,b] \times [c,d]$ and $f : R \to \mathbb{R}$ be defined by $f(x,y) = p(x)q(y)$ where $p : [a,b] \to \mathbb{R}$ and $q : [c,d] \to \mathbb{R}$ are continuous. Show that $\iint\limits_R f(x,y)dxdy = \left(\int\limits_a^b p(x)dx\right)\left(\int\limits_c^d q(y)dy\right)$.

2. Let $R = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$. Evaluate $\iint\limits_R \sin x \cos y dxdy$.

3. Evaluate $\iint\limits_R \cos x^3 dxdy$ where $R$ is the region in $\mathbb{R}^2$ bounded by $y = 3x^2, y = 0$ and $x = 1$.

4. Let $R$ be the region lying below the curve $y = \cos x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and above the x-axis. Evaluate $\iint\limits_R \sin x dxdy$.

5. Let $R$ be the region in $\mathbb{R}^2$ bounded by the curves $y = 2x^2$ and $y = 1 + x^2$. Evaluate $\iint\limits_R (2x^2 + y)dxdy$.

6. Evaluate $\iint\limits_R x\cos(y - \frac{y^3}{3})dxdy$ where $R = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le 1, x \ge 0, y \ge 0\}$.

7. Evaluate the following iterated integrals by interchanging the order of integration.

   (a) $\int\limits_0^1 \int\limits_y^1 \cos x^2 dxdy$.

   (b) $\int\limits_0^1 \int\limits_{\sqrt{x}}^1 e^{y^3} dydx$.

   (c) $\int\limits_0^1 \int\limits_{x^2}^1 x^3 e^{y^3} dydx$.

   (d) $\int\limits_0^1 \int\limits_y^1 \frac{1}{1+x^4} dxdy$.

8. Evaluate $\int\limits_0^1 (\tan^{-1}\pi x - \tan^{-1} x)dx$.

9. Find the volume of the solid enclosed by the surfaces $z = 6 - x^2 - y^2$, $z = 2x^2 + y^2 - 1$, $x = -1$, $x = 1$ , $y = -1$ and $y = 1$.

10. Let $D$ be the solid bounded by the surfaces $y = x^2, y = 3x, z = 0$ and $z = x^2 + y^2$. Find the volume of $D$.

11. Led $D$ be the solid bounded by the cylinder $x^2 + y^2 = 1$ and the planes $y + z = 1$ and $z = 0$. Find the volume of $D$.

12. Find the volume of the solid which is common to the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$. See Problem 10 of PP 20.

## Practice Problems 32: Hints/Solutions

1. Follows from the Fubini's theorem.

2. By Fubuni's theorem $\iint\limits_R \sin x \cos y dxdy = (\int\limits_0^{\frac{\pi}{2}} \sin x dx)(\int\limits_0^{\frac{\pi}{2}} \cos y dy) = 1$.

3. See Figure 1. By Fubuni's theorem $\iint\limits_R \cos x^3 dxdy = \int\limits_0^1 \int\limits_0^{3x^2} \cos x^3 dydx = \int\limits_0^1 3x^2 \cos x^3 dx = \sin 1$.

4. See Figure 2. $\iint\limits_R \sin x dxdy = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int\limits_0^{\cos x} \sin x dydx = \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \cos x dx = \frac{\sin^2 x}{2}|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 0$.

5. See Figure 3. $\iint\limits_R (2x^2 + y) dxdy = \int\limits_{-1}^{1} \int\limits_{2x^2}^{1+x^2} (2x^2 + y) dydx$.

6. See Figure 4. $\iint\limits_R x \cos(y - \frac{y^3}{3}) dxdy = \int\limits_0^1 \int\limits_0^{\sqrt{1-y^2}} x \cos(y - \frac{y^3}{3}) dxdy = \int\limits_0^1 \frac{1}{2} x^2 \cos(y - \frac{y^3}{3})|_0^{\sqrt{1-y^2}} = \frac{1}{2} \int\limits_0^1 (1 - y^2) \cos(y - \frac{y^3}{3}) dy = \frac{1}{2} \int\limits_0^{\frac{2}{3}} \cos t dt$.

7. (a) $\int\limits_0^1 \int\limits_y^1 \cos x^2 dxdy = \int\limits_0^1 \int\limits_0^x \cos x^2 dydx = \int\limits_0^1 x \cos x^2 dx = \frac{1}{2} \sin 1$. See Figure 5.

   (b) $\int\limits_0^1 \int\limits_{\sqrt{x}}^1 e^{y^3} dydx = \int\limits_0^1 \int\limits_0^{y^2} e^{y^3} dxdy = \frac{1}{3} \int\limits_0^1 e^u du = \frac{1}{3}(e - 1)$. See Figure 6.

   (c) $\int\limits_0^1 \int\limits_{x^2}^1 x^3 e^{y^3} dydx = \int\limits_0^1 \int\limits_0^{\sqrt{y}} x^3 e^{y^3} dxdy = \int\limits_0^1 \frac{1}{4} y^2 e^{y^3} dy = \frac{1}{12}(e - 1)$. See Figure 7.

   (d) $\int\limits_0^1 \int\limits_y^1 \frac{1}{1+x^4} dxdy = \int\limits_0^1 \int\limits_0^x \frac{1}{1+x^4} dydx = \frac{1}{2} \int\limits_0^1 \frac{du}{1+u^2} = \frac{1}{2} \tan^{-1} u|_0^1 = \frac{\pi}{8}$. See Figure 8.

8. Note that $\int\limits_0^1 (\tan^{-1} \pi x - \tan^{-1} x) dx = \int\limits_0^1 \int\limits_x^{\pi x} \frac{1}{1+y^2} dydx = \int\limits_0^1 \int\limits_{\frac{y}{\pi}}^{y} \frac{1}{1+y^2} dxdy + \int\limits_1^{\pi} \int\limits_{\frac{y}{\pi}}^{1} \frac{1}{1+y^2} dxdy$. See Figure 9.

9. Note that $(6 - x^2 - y^2) - (2x^2 + y^2 - 1) \geq 0$ for all $(x, y) \in [-1, 1] \times [-1, 1]$. The volume of $D = \int\limits_{-1}^{1} \int\limits_{-1}^{1} (6 - x^2 - y^2) - (2x^2 + y^2 - 1) dydx$.

10. Let $R$ be the region in $\mathbb{R}^2$ bounded by the curves $y = x^2$ and $y = 3x$. Then the volume of $D = \iint\limits_R (x^2 + y^2) dxdy = \int\limits_0^3 \int\limits_{x^2}^{3x} (x^2 + y^2) dydx$.

11. Let $R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. Then the solid $D$ lies above the region $R$ and below the graph $z = 1 - y$. The volume of $D = \iint\limits_R (1 - y) dxdy = \iint\limits_R dxdy - \iint\limits_R ydxdy$. Note that $\iint\limits_R ydxdy = \int\limits_{-1}^{1} \int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} ydydx = \int\limits_{-1}^{1} 0 dx = 0$. Therefore the required volume is the area of $R$ which is $\pi$.

12. The solid is enclosed by the cylinder $x^2 + y^2 = 1$ and the surfaces $z = -\sqrt{1 - x^2}$ and $z = \sqrt{1 - x^2}$. Let $R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$. The required volume is equal to $\iint\limits_R (\sqrt{1 - x^2} - (-\sqrt{1 - x^2})) = \int\limits_{-1}^{1} \int\limits_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2\sqrt{1 - x^2} = \frac{16}{3}$.