PP 33 : Change of variables in double integrals, Polar coordinates

1. Consider the transformation $T : [0, 2\pi] \times [0, 1] \to \mathbb{R}^2$ given by $T(u, v) = (2v \cos u, v \sin u)$.

   (a) For a fixed $v_0 \in [0, 1]$, describe the set $\{T(u, v_0) : u \in [0, 2\pi]\}$.

   (b) Describe the set $\{T(u, v) : (u, v) \in [0, 2\pi] \times [0, 1]\}$.

2. Let R be the region in $\mathbb{R}^2$ bounded by the straight lines $y = x, y = 3x$ and $x + y = 4$. Consider the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(u, v) = (u - v, u + v)$. Find the set $S \subset \mathbb{R}^2$ satisfying $T(S) = R$.

3. Let R be the region in $\mathbb{R}^2$ bounded by the curve defined in the polar co-ordinates $r = 1 - \cos\theta, 0 \le \theta \le \pi$ and the x-axis. Consider the transformation $T : [0, \pi] \times [0, 1] \to \mathbb{R}^2$ defined by $T(r, \theta) = (r \cos\theta, r \sin\theta)$. Let S be the subset of $[0, \pi] \times [0, 1]$ satisfying $T(S) = R$. Sketch the regions S and R.

4. Using the change of variables $u = x + y$ and $v = x - y$, show that $\int_0^1 \int_0^x (x - y)dydx = \int_0^1 \int_v^{2-v} \frac{v}{2} dudv$.

5. Let R be the region bounded by $x = 0, x = 1, y = x$ and $y = x+1$. Show that $\iint_R \frac{dxdy}{\sqrt{xy - x^2}} = \left(\int_0^1 \frac{du}{\sqrt{u}}\right)\left(\int_0^1 \frac{dv}{\sqrt{v}}\right)$.

6. Show that $\int_0^1 \int_0^{1-x} e^{\frac{x-y}{x+y}} dxdy = \frac{1}{2}\int_0^1 \int_{-v}^{v} e^{\frac{u}{v}} dudv = \frac{1}{2}\sinh(1)$.

7. Find the region R in $\mathbb{R}^2$ satisfying $\int_{\frac{1}{\sqrt{2}}}^{1} \int_{\sqrt{1-x^2}}^{x} xydydx + \int_1^{\sqrt{2}} \int_0^x xydydx + \int_{\sqrt{2}}^{2} \int_0^{\sqrt{4-x^2}} xydydx = \iint_R xydxdy$. Evaluate $\iint_R xydxdy$.

8. Convert $\int_0^1 \int_{x^2}^{x} dydx$ in to an iterated integral involving polar coordinates.

9. Evaluate

   (a) $\int_0^1 \int_0^{1-y} \sqrt{x + y}(y - 2x)^2 dxdy$.

   (b) $\int_0^{\frac{1}{\sqrt{2}}} \int_y^{\sqrt{1-y^2}} (x + y)dxdy$.

   (c) $\int_1^2 \int_0^y \frac{1}{(x^2+y^2)^{\frac{3}{2}}} dxdy$.

   (d) $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2 + y^2} dydx$.

10. Let $R = \{(x, y) \in \mathbb{R}^2 : 9x^2 + 4y^2 \le 1\}$. Evaluate $\iint_R \cos(9x^2 + 4y^2)dxdy$.

11. Find the volume of the solid bounded by the surfaces $z = 3(x^2 + y^2)$ and $z = 4 - (x^2 + y^2)$.

12. Find the volume of the solid in the first octant bounded below by the surface $z = \sqrt{x^2+y^2}$ and above by $x^2+y^2+z^2 = 8$ as well as the planes $y = 0$ and $y = x$.

Practice Problems 33: Hints/Solutions

1. (a) If $x = 2v_0 \cos u$ and $y = v_0 \sin u$ then $\frac{x^2}{4} + \frac{y^2}{1} = v_0^2$. The set $\{T(u, v_0) : u \in [0, 2\pi]\}$ is an ellipse.

   (b) The set is the region enclosed by the ellipse $\frac{x^2}{4} + \frac{y^2}{1} = 1$.

2. If $x = u - v$ and $y = u + v$ then $y = x \Rightarrow v = 0$, $y = 3x \Rightarrow v = \frac{u}{2}$ and $x + y = 4 \Rightarrow u = 2$. The region S is bounded by the lines $v = 0, v = \frac{u}{2}$ and $u = 2$ in the uv-plane. See Figure 1.

3. See Figure 2.

4. Note that $\int\limits_0^1 \int\limits_0^x (x-y)dydx = \iint\limits_R (y-x)dxdy$ where R is the region in xy-plane bounded by the lines $y = x$, $x = 1$ and $y = 0$. Since $x = \frac{1}{2}(u+v)$ and $y = \frac{1}{2}(u-v)$, $y = 0 \Rightarrow u = v$, $x = 1 \Rightarrow u + v = 2$ and $x = y \Rightarrow v = 0$. Therefore $\iint\limits_R (y-x)dxdy = \iint\limits_S v\frac{\partial(x,y)}{\partial(u,v)}dudv$ where S is the region in the uv-plane bounded by the lines $u = v$, $v + v = 2$ and $v = 0$.

5. Take $u = x$ and $v = y - x$. Then $y = x \Rightarrow v = 0$ and $y = x + 1 \Rightarrow v = 1$. Therefore $\iint\limits_R \frac{dxdy}{\sqrt{xy-x^2}} = \iint\limits_S \frac{1}{\sqrt{uv}}\frac{\partial(x,y)}{\partial(u,v)}dudv$ where S is the region in the uv-plane bounded by the lines $u = 0$, $u = 1$, $v = 0$ and $v = 1$.

6. Consider $u = x - y$ and $v = x + y$. Then $\int\limits_0^1 \int\limits_0^{1-x} e^{\frac{x-y}{x+y}}dxdy = \iint\limits_S e^{\frac{u}{v}}\frac{\partial(x,y)}{\partial(u,v)}dudv$ where S is the region in the $uv-$plane bounded by the lines $u = -v$, $u = v$ and $v = 1$.

7. See Figure 3. By polar coordinates, $\iint_D xydxdy = \int\limits_0^{\frac{\pi}{4}} \int\limits_1^2 r^3 \cos\theta \sin\theta drd\theta = \frac{15}{4}\int\limits_0^{\frac{\pi}{4}} \sin\theta\cos\theta d\theta$.

8. The integral becomes $\iint\limits_D dxdy$ where D is the region in the first quadrant in $\mathbb{R}^2$ bounded by the line $y = x$ and the curve $y = x^2$. The equation $y = x^2$ can be converted in polar as $r\sin\theta = r^2\cos^2\theta$ which implies $r = \tan\theta\sec\theta$. Therefore $\iint\limits_D dxdy = \int\limits_0^{\frac{\pi}{4}} \int\limits_0^{\sec\theta\tan\theta} rdrd\theta$.

9. (a) Note that $\int\limits_0^1 \int\limits_0^{1-y} \sqrt{x+y}(y-2x)^2dxdy = \iint\limits_R \sqrt{x+y}(y-2x)^2dxdy$ where R is the region bounded by the lines $x = 0, y = 0$ and $x + y = 1$. Consider $u = x + y$ and $v = y - 2x$. Then $x = 0 \Rightarrow v = u$, $y = 0 \Rightarrow v = -2u$ and $x + y = 1 \Rightarrow u = 1$. Therefore $\iint\limits_R \sqrt{x+y}(y-2x)^2dxdy = \int\limits_0^1 \int\limits_{-2u}^{u} \sqrt{u}v^2\frac{1}{3}dvdu$.

   (b) The given integral becomes $\iint\limits_R (x+y)dxdy$ where R is the region bounded by the lines $y = 0, y = x$ and the circle $x^2+y^2 = 1$. By polar coordinates $\iint\limits_R (x+y)dxdy = \int\limits_0^{\frac{\pi}{4}} \int\limits_0^1 (r\cos\theta + r\sin\theta)rdrd\theta$.

(c) See Figure 4. The given integral becomes $\int_0^{\frac{\pi}{4}} \int_{\sec\theta}^{2\sec\theta} \frac{1}{r^3} r dr d\theta$.

(d) See Figure 5. The given integral becomes $\iint_R \sqrt{x^2+y^2} dxdy$ where R is the region in the first quadrant bounded by the circle $(x-1)^2+y^2=1$ and the x-axis. The points on the circle $y^2=2x-x^2$ is represented by $r=2\cos\theta$ in polar coordinates. Therefore the integral is given by $\int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} r r dr d\theta$.

10. Take $x=\frac{r}{3}\cos\theta$ and $y=\frac{r}{2}\sin\theta$. Then $\frac{\partial(x,y)}{\partial(r,\theta} = \frac{r}{6}$. Therefore $\iint_R \cos(9x^2+4y^2) dxdy = \int_0^{2\pi} \int_0^1 \cos(r^2) \frac{r}{6} dr d\theta = \int_0^{2\pi} \int_0^1 \cos u \frac{du}{12} d\theta$.

11. The intersection of the surfaces is the set $\{(x,y,3) : x^2+y^2=1\}$. Therefore the volume is given by $\iint_R (4-x^2-y^2-3(x^2+y^2)) dxdy$ where R is the region in $\mathbb{R}^2$ enclosed by the circle $x^2+y^2=1$. By polar coordinate the integral becomes $\int_0^{2\pi} \int_0^1 (4-4r^2) r dr d\theta$.

12. The given solid lies above the region R where R is in the first quadrant in $\mathbb{R}^2$ bounded by the circle $x^2+y^2=4$ and the lines $y=x$ and $y=0$. Therefore the required volume is given by $\iint_R (\sqrt{8-x^2-y^2} - \sqrt{x^2+y^2}) dxdy = \int_0^{\frac{\pi}{4}} \int_0^2 (\sqrt{8-r^2} - r) r dr d\theta$.