PP 36 : Line integrals

1. Let $f(x,y) = (xy, y^2)$. Evaluate $\int_{C_i} f \cdot dR$ where $C_i, i = 1, 2$ and 3 are given as follows.

   (a) C_1 is the upper half of the unit circle traversing from $(-1, 0)$ and $(1, 0)$.

   (b) C_2 is the part of the curve $y = x^2$ traversing from $(0, 0)$ to $(1, 1)$.

   (c) C_3 is the part of the curve $y = x$ traversing from $(0, 0)$ to $(1, 1)$.

2. Let C be a circular arc connecting $(0, 0, 0)$ and $(1, 1, 1)$. Show that

   (a) $\int_C y^2z^3dx + 2xz^3ydy + 3xy^2z^2dz = 1$.

   (b) $\int_C 3x^2dx + 2yzdy + y^2dz = 2$.

3. Let $F(x, y, z) = (2xy^2 + 3x^2, 2yx^2, 1)$. Evaluate $\int_{C_i} F \cdot dR$ where $C_i, i = 1, 2$ and 3 are given as follows.

   (a) C_1 is the curve $(t^9, \sin^9(\frac{\pi t}{2}), t), 0 \leq t \leq 1$.

   (b) C_2 is a circular arc connecting $(0, 0, 0)$ and $(1, 1, 1)$.

   (c) C_3 is the curve obtained by intersecting the surfaces $x^2 + y^2 = 1$ and $z = x^2 + y^2$.

4. Let C be a curve represented by two parametric representations such that $C = \{R_1(s) : s \in [a, b]\} = \{R_2(t) : t \in [c, d]\}$ where $R_1 : [a, b] \to \mathbb{R}^3$ and $R_2 : [c, d] \to \mathbb{R}^3$ be two distinct differentiable one-one maps.

   (a) Show that there exists a function $h : [c, d] \to [a, b]$ such that $R_2(t) = R_1(h(t))$.

   (b) i. If R_1 and R_2 trace out C in the same direction then $\int_C f \cdot dR_1 = \int_C f \cdot dR_2$.
   ii. If R_1 and R_2 trace out C in opposite direction then $\int_C f \cdot dR_1 = -\int_C f \cdot dR_2$.

5. Consider the curve C which is the intersection of the surfaces $x^2 + y^2 = 1$ and $z = x^2$. Assume that C is oriented counterclockwise as seen from the positive $z-$axis. Evaluate $\int_C zdx - xydy - xdz$.

6. Let $f = (f_1, f_2, f_3)$ where $f_i : \mathbb{R}^3 \to \mathbb{R}, i = 1, 2, 3$. Suppose the partial derivatives of f_1, f_2 and f_3 are continuous. If there exists ϕ such that $\bigtriangledown\phi = f$ then show that

   (a) $\frac{\partial f_2}{\partial x} = \frac{\partial f_1}{\partial y}, \frac{\partial f_3}{\partial y} = \frac{\partial f_2}{\partial z}$ and $\frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x}$;

   (b) $curlf = 0$ where $curlf = (\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z})i + (\frac{\partial f_1}{\partial z} = \frac{\partial f_3}{\partial x})j + (\frac{\partial f_2}{\partial x} = \frac{\partial f_1}{\partial y})k$.

7. Let $f(x, y, z) = (x^2, xy, 1)$. Show that that there is no ϕ such that $\bigtriangledown\phi = f$.

8. Let $S = \{(x, y) : (x, y) \neq (0, 0)\}$ and $f(x, y) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})$ for $(x, y) \in S$.

   (a) If C is the unit circle then $\int_C f \cdot dR = 0$.

   (b) Verify that $\bigtriangledown\phi = f$ where $\phi(x, y) = \ln(\sqrt{x^2 + y^2}), (x, y) \in S$.

9. Let $S = \{(x, y) : (x, y) \neq (0, 0)\}$ and $f(x, y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ for $(x, y) \in S$.

   (a) Show that f satisfies the condition given in Problem 6(a); that is, $\frac{\partial}{\partial y}(\frac{-y}{x^2+y^2}) = \frac{\partial}{\partial x}(\frac{x}{x^2+y^2})$ for $(x, y) \in S$.

   (b) If C is the unit circle then $\int_C f \cdot dR \neq 0$.

   (c) Show that there is no ϕ on S such that $\bigtriangledown\phi = f$.

(In the next lecture we will see that if a circle C and its interior lie entirely in S then $\int_C f \cdot dR = 0$.)

Practice Problems 36: Hints/Solutions

1. (a) Consider the parametrization $R(t) = (\sin t, \cos t), \frac{3\pi}{2} \le t \le \frac{5\pi}{2}$ to describe C_1. Then $\int_{C_1} f \cdot dR = \int_{\frac{3\pi}{2}}^{\frac{5\pi}{2}} (\sin t \cos t, \cos^2 t)(\cos t, -\sin t)dt = 0$. (Alternatively, we can also consider the parametrization $R(t) = (\cos t, \sin t), 0 \le t \le \pi$. In this case the curve traverse from $(1, 0)$ to $(-1, 0)$ and $\int_{C_1} f \cdot dR = -\int_0^{\pi} (\sin t \cos t, \sin^2 t)(-\sin t, \cos t)dt = 0$. See Problem 4(a) ii.)

   (b) The curve C_2 can be parametrized as $R(x) = (x, x^2), 0 \le x \le 1$. Therefore $\int_{C_2} f \cdot dR = \int_0^1 (x^3, x^4) \cdot (1, 2x)dx = \frac{7}{12}$.

   (c) Considering $R(x) = (x, x), 0 \le x \le 1$ we have $\int_{C_3} f \cdot dR = \int_0^1 (x^2, x^2) \cdot (1, 1)dx = \frac{2}{3}$.

2. (a) Note that C could be any circular arc and the value of the line integral has to be independant of the arc to be considered. Observe that the given integral is $\int_C \nabla f \cdot dR$ for $f(x, y, z) = xy^2z^3$. The value of the required integral is $f(1, 1, 1) - f(0, 0, 0)$.

   (b) If $\nabla f(x, y, z) = (3x^2, 2yz, y^2)$ for some f, then $f = x^3 + g(y, z)$ for some g. Since $f_y = g_y = 2yz$, $g = y^2z + h(z)$ for some h. Hence $f = x^3 + y^2z + h(z)$. Since $f_z = y^2$, we can consider $f = x^3 + y^2z$. The value of the required integral is $f(1, 1, 1) - f(0, 0, 0)$.

3. Observe that $\nabla f = F$ where $f(x, y, z) = x^2y^2 + x^3 + z$.

   (a) The value of the integral is $f(1, 1, 1) - f(0, 0, 0) = 3$.

   (b) The value of the integral is $f(1, 1, 1) - f(0, 0, 0)$.

   (c) Since the curve is closed the value of the integral is 0.

4. (a) Consider $h(t) = R_1^{-1}(R_2(t))$.

   (b) By chain rule $R_2'(t) = R_1'(h(t)h'(t)$. Therefore $\int_C f \cdot dR_2 = \int_c^d f(R_2(t)) \cdot R_2'(t)dt = \int_c^d f(R_1(h(t))) \cdot R_1'(h(t))h'(t)dt$. Changing variable from t to u implies that $\int_C f \cdot dR_2 = \int_{h(c)}^{h(d)} f(R_1(u)) \cdot R_1'(u)du = \pm \int_a^b f(R_1(u)) \cdot R_1'(u)du = \pm \int_C f \cdot dR_1$.

5. The cylinder $x^2 + y^2 = 1$ is parametrized as $(\cos\theta, \sin\theta, z), 0 \le \theta \le 2\pi, z \in \mathbb{R}$. Since C also lies in $z = x^2$, the curve is parametrized as $R(\theta) = (\cos\theta, \sin\theta, \cos^2\theta), 0 \le \theta \le 2\pi$. The required integral is $\int_0^{2\pi} \{\cos^2\theta(-\sin\theta) - \cos^2\theta(\sin\theta) - \cos\theta(-2\sin\theta\cos\theta)\}d\theta = 0$.

6. (a) Let $\nabla\phi = f$. Then $\frac{\partial f_2}{\partial x} = \frac{\partial}{\partial x}(\frac{\partial \phi}{\partial y}) = \frac{\partial}{\partial y}(\frac{\partial \phi}{\partial x}) = \frac{\partial f_1}{\partial y}$.

   (b) Follows from (a).

7. Here $f_1(x, y, z) = x^2, f_2(x, y, z)) = xy$ and $f_3(x, y, z) = 1$. Apply Problem 6.

8. Both (a) and (b) can be easily verified.

9. (a) Easy to verify.

   (b) Verify that $\int_C f \cdot dR = 2\pi$ (See Problem 1 in the notes).

   (c) If so then by the second FTC of line integral, $\int_C f \cdot dR = 0$ which is not so by (b).