PP 37 : Green's Theorem

The plane curve C described in this problem sheet is oriented counterclockwise.

1. Evaluate the line integral

$$\oint_C (x^2 \sin^2 x - y^3)dx + (y^2 \cos^2 y - y)dy$$

   where C is the closed curve consisting of $x + y = 0$, $x^2 + y^2 = 25$ and $y = x$ and lying in the first and fourth quadrant.

2. Let a square R be enclosed by C and

$$\oint_C (xy^2 + x^3 \sin^3 x)dx + (x^2 y + 2x)dy = 6.$$

   Find the area of the square.

3. Let C be a simple closed smooth curve and α be a real number. Suppose

$$\oint_C (\alpha e^x y + e^x)dx + (e^x + ye^y)dy = 0.$$

   Find α.

4. Let D be the region enclosed by a simple closed piecewise smooth curve C. Let F, F_x and F_y be continuous on an open set containing D. Show that

$$\iint_D F_x dxdy = \oint_C F dy \quad \text{and} \quad \iint_D F_y dxdy = -\oint_C F dx.$$

5. Let C be the ellipse $x^2 + xy + y^2 = 1$. Evaluate $\oint_C (\sin y + x^2)dx + (x \cos y + y^2)dy$.

6. Let D be the region enclosed by a simple closed smooth curve C. Show that

$$\text{Area of } D = \oint_C xdy = -\oint_C ydx.$$

7. Evaluate the area of the region enclosed by the simple closed curve $x^{2/3} + y^{2/3} = 1$.

8. Find the area between the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ and the circle $x^2 + y^2 = 25$.

9. Let $f : [a, b] \to \mathbb{R}$ be a non-negative function such that its first derivative is continuous. Suppose C is the boundary of the region bounded above by the graph of f, below by the x-axis and on the sides by the lines $x = a$ and $x = b$. Show that

$$\int_a^b f(x)dx = -\oint_C ydx.$$

10. Let D be the region enclosed by the rays $\theta = a$, $\theta = b$ and the curve $r = f(\theta)$. Use Green's theorem to derive the formula

$$A = \frac{1}{2}\int_a^b r^2 d\theta$$

    for the area of D.

Practice Problems 37: Hints/Solutions

1. Let R be the region enclosed by C (see Figure 1). By Green's theorem

$$\oint_C (x^2\sin^2 x - y^3)dx + (y^2\cos^2 y - y)dy = \iint_R 3y^2 dxdy = \int_{-\pi/4}^{\pi/4}\int_0^5 3r^3\sin^2\theta dr d\theta.$$

2. By Green's theorem,

$$\oint_C (xy^2 + x^3\sin^3 x)dx + (x^2y + 2x)dy = \iint_R 2 = 6.$$

The area of R is 3.

3. Let R be the region enclosed by C. By Green's theorem,

$$\oint_C (\alpha e^x y + e^x)dx + (e^x + ye^y)dy = \iint_R (1-\alpha)e^x dxdy = 0.$$

Hence $\alpha = 1$.

4. Follows from Green's theorem.

5. By Green's theorem, $\oint_C (\sin y + x^2)dx + (x\cos y + y^2) = 0$.

6. Follows from Green's theorem.

7. Let C denote the curve (see Figure 2). Then C is parameterized as $x(\theta) = \cos^3\theta$ and $y(\theta) = \sin^3\theta$, $0 \le \theta \le 2\pi$. The required area is

$$A = \frac{1}{2}\oint_C xdy - ydx = \frac{3}{2}\int_0^{2\pi}\sin^2\theta\cos^2\theta d\theta = \frac{3}{8}\int_0^{2\pi}\sin^2 2\theta d\theta = \frac{3}{8}\pi.$$

8. The circle and the ellipse are parameterized as

$$x_1(\theta) = (5\cos\theta, 5\sin\theta) \quad \text{and} \quad x_2(\theta) = (2\sin\theta, 3\cos\theta), \quad \theta \in [0, 2\pi],$$

(see Figure 3). The require area is $A = \frac{1}{2}\int_0^{2\pi}[(5\cos\theta)(5\cos\theta) + (5\sin\theta)(5\sin\theta)]d\theta$ $+\frac{1}{2}\int_0^{2\pi}[-(2\sin\theta)(3\sin\theta) + (-3\cos\theta)(2\cos\theta)]d\theta = 19\pi$.

9. Follows from Problem 6 (See Figure 4).

10. Parameterize the rays and the curve as follows (see Figure 5):

$$C_1 := (r\cos a, r\sin a), \quad 0 \le r \le f(a),$$

$$C_2 := (r(\theta)\cos\theta, r(\theta)\sin\theta), \quad a \le \theta \le b,$$

$$C_3 := (r\cos b, r\sin b), \quad 0 \le r \le f(b).$$

The required area is $\frac{1}{2}(\oint_{C_1} + \oint_{C_2} - \oint_{C_3})\{xdy - ydx\}$.