Math 301 Tutorial 10

1. Reimann-Stieljes Integral

(1) Let $\alpha_1, \alpha_2, \alpha : [a,b] \to \mathbb{R}$ be increasing functions. Let $f, g : [a,b] \to \mathbb{R}$ be integrable with respect to α. Let $c \in \mathbb{R}$. Show that

(a) $cf \in R_\alpha[a,b]$ and $\int_a^b cf \, \mathrm{d}\alpha = c\int_a^b f \, \mathrm{d}\alpha$.

(b) If $f \in R_{\alpha_1}[a,b]$ and $f \in \mathbb{R}_{\alpha_2}[a,b]$ then $f \in R_{\alpha_1+\alpha_2}[a,b]$ and

$$\int_a^b f \, \mathrm{d}(\alpha_1 + \alpha_2) = \int_a^b f \, \mathrm{d}\alpha_1 + \int_a^b f \, \mathrm{d}\alpha_2$$

(c) $f \in R_{c\alpha}[a,b]$ and

$$\int_a^b f \, \mathrm{d}(c\alpha) = \int_a^b f \, c\mathrm{d}\alpha.$$

(d) If $f \leq g$ then

$$\int_a^b f \, \mathrm{d}\alpha \leq \int_a^b g \, \mathrm{d}\alpha.$$

(2) Page 218, Problem 6(a)

(3) Page 225, Problem 26.

(4) Page 234, Problem 50.

(5) Give an example of a bounded real function on $[0,1]$ which is not integrable.

2. Equicontinuity and Ascoli-Arzela Theorem

Definition 1. A set $\mathscr{F} \subset \mathrm{B}(X)$ of real valued functions on (X,d), is said to be **uniformly bounded** if $\sup_{f\in\mathscr{F}} \|f\|_\infty < \infty$.

Definition 2. A set $\mathscr{F} \subset \mathrm{B}(X)$ of real valued functions on (X,d) is called **equicontinuous** if given any $\epsilon > 0$ there exists a $\delta > 0$ such that

$$d(x,y) < \delta \implies |f(x) - f(y) < \epsilon, \ \forall f \in \mathscr{F}.$$

(1) Show that every totally bounded subset of $\mathrm{C}(X)$ is uniformly bounded.

(2) Show that every totally bounded subset of $\mathrm{C}(X)$ is equicontinuous. (Hint: Every totally bounded set can be covered by finite number of ϵ-balls)

(3) Let (f_n) be a convergent sequence in $\mathrm{C}(X)$. Then $\{f_n \mid n \in \mathbb{N}\}$ is uniformly bounded and equicontinuous. (Show $\{f_n \mid n \in \mathbb{N}\}$ is totally bounded.)

(4) Let (X, d) be a compact metric space and $\mathscr{F} \subset \mathrm{C}(X)$. Show that if $\mathscr{F}$ is closed then it is complete.

(5) Let (X, d) be a compact metric space and $\mathscr{F} \subset \mathrm{C}(X)$. Let $\mathscr{F}$ be closed uniformly bounded and equicontinuous.

(a) Show that any sequence (f_n) in $\mathrm{C}(X)$ is equicontinuous and uniformly bounded.

(b) Since X is totally bounded, there exist x_i such that each $d(x, x_i) < \delta$ for some i and each $x \in X$. Using that (f_n) is uniformly bounded show that there exists a subsequence (f_{n_k}) such that we can find $\mathbb{N}$,

$$|f_{n_k} - f_{n_l}| < \epsilon/3, \text{ for any } l, k \geq N.$$

for all i.

(c) Using the previous two facts show that f_n has a Cauchy subsequence.

(d) Conclude that $\mathscr{F}$ is compact if and only if $\mathscr{F}$ is closed, uniformly bounded and equicontinuous. This is the Ascoli-Arzela theorem