Tutorial II

1. Let A and B be bounded non empty subsets of $\mathbb{R}$
   (a) Show that $A \cup B$ is a bounded subset of $\mathbb{R}$. Futher prove that
   $$\sup\,(A \cup B) = \sup\,\{\sup\,A, \sup\,B\}.$$
   (b) Let $A + B = \{a + b \mid a \in A, b \in B\}$. Show that $\inf\,(A + B) = \inf\,A + \inf\,B$.

2. Show that every Cauchy sequence in $\mathbb{R}$ is bounded.

3. (Prob. 11, pg7) Fix $a > 0$ and let $x_1 > \sqrt{a}$. For $n \geq 1$, define
   $$x_{n+1} = \frac{1}{2}\Big(x_n + \frac{a}{x_n}\Big).$$
   Show that (x_n) converges and $\lim\,_{n\to\infty} x_n = \sqrt{a}$.

4. (Prob 18, pg 10)
   (a) Given $a > -1$, $a \neq 0$ use induction to show that $(1 + a)^n > 1 + na$ for any integer $n > 1$.
   (b) Use (a) to show that, for any $x > 0$ the sequence $(1 + \frac{x}{n})^n$ increases.
   (c) If $a > 0$, show that $(1 + a)^r > 1 + ra$ holds for any rational exponent $r > 1$. (Hint : If $r = p/q$ then apply (a) with $n = q$ and (b) with $x = ap$.]
   (d) Finally show that (c) holds for any *real* exponent $r > 1$.

5. (Prob 19, pg 10) If $0 < c < 1$ show that $c^n \to 0$; and if $c > 0$ show that $c^{1/n} \to 1$. [Hint: Use Bernoullis' inequality in each case, once with $c = 1/(1 + x),\ x > 0$ and once with $c^{1/n} = 1 + x_n$, where $x_n > 0$]

6. If $r \in \mathbb{Q}$ and x is irrational then show that $r + x$ and rx are irrational. (Hint: Prove by contradiction)

7. (Prob 15, pg 7) Show that a Cauchy sequence of real numbers with a convergent subsequence converges.