MTH 301 Tutorial III
Due on August 16, 2011
Submit only the starred exercises

1. Problem 11 in the textbook, Page 21. Look in the book for details.
   Let $a_1 = 0$ and for $n = 2, 3, \cdots$, let $a_n = \sum_{i=1}^{n-1} i = n(n-1)/2$. Show that the correspondence $(m, n) \to a_{m+n-1} + n$ from $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection.

2. Show that $\mathbb{Q}$ is countable and $\mathbb{R} - \mathbb{Q}$ is uncountable.

3.* (i) Show that if $A_1, \cdots, A_k$ are finite sets then $\cup_{i=1}^{k} A_i$ is finite. (Prove by definition.)

   (ii) What can you say about countably infinite union of finite sets, is it finite? If yes, give a proof. If not give a counterexample.

   (iii) Show that if $A_1, A_2, \cdots, A_n$ are countable sets then $A_1 \times A_2 \times \cdots \times A_n$ is countable.

4.* Problem 16 in the book, Page 22. Look in the book for details.
   The algebraic numbers are those real or complex numbers that are the roots of polynomials having integer coefficients. Prove that the set of algebraic numbers is countable. [Hint: First show that the set of polynomials having integer coefficients is countable.]

5.* Problem 25, page 29
   Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = 1$ if $x \in \Delta$ and $g(x) = 0$ otherwise. At which points of $\mathbb{R}$ is g continuous?

6. Show that the following statements are equivalent.
   (i) A is a countable set.
   (ii) There exists a one-one function from $A \to \mathbb{N}$.
   (iii) There exists an onto function from $\mathbb{N} \to A$.

7.* Let B be a uncountable set and $f : A \to B$ be an onto function.
   (i) Show that A is uncountable.
   (ii) If f is one-one everywhere except on a countable set, show that there exists a bijection from A to B, that is, they are equivalent.

8.* Prove Corollary 2.18 in the textbook, page 32.

9.* Problem 24, Page 29 Show that Δ is perfect; that is, every point in Δ is the limit of a sequence of distinct points from Δ. In fact show that every point in Δ is the limit of a sequence of distinct endpoints.

10. Problem 26, Page 29

    Let $f : \Delta \to [0, 1]$ be the Cantor function and let $x, y \in \Delta$ with $x < y$. Then show that $f(x) \leq f(y)$. If $f(x) = f(y)$ show that $f(x)$ has two distinct binary expansions. Finally, show that $f(x) = f(y)$ if and only if x and y are "consecutive" endpoints of the form $x = 0.a_1a_2 \cdots a_n1$ and $y = 0.a_1a_2 \cdots a_n2$. (Read proof of Corollary 2.16, to first understand why f is not one-one.)