MTH 301 TUTORIAL IV

DUE ON AUGUST 30, 2011

SUBMIT ONLY THE STARRED EXERCISES

1. Show that any collection of pairwise, disjoint nonempty open intervals in $\mathbb{R}$ is at most countable.

2. Let $2^{\mathbb{N}}$ denote the set of all sequences of 0's and 1's. Show that $d(a,b) = \Sigma_{n=1}^{\infty} 2^{-n} \mid a_n - b_n \mid$ where $(a_n), (b_n) \in 2^{\mathbb{N}}$ is a metric on $2^{\mathbb{N}}$.

3. * Check which of the following given functions $d : \mathbb{R} \times \mathbb{R} \to R$ are metrics on $\mathbb{R}$. If yes, show that its a metric, else explain why it is not a metric.
   (i) $d(x,y) = (x-y)^2$.

   (ii) $d(x,y) = \sqrt{\mid x-y \mid}$.

   (iii) $d(x,y) = \dfrac{\mid x-y \mid}{1+ \mid x-y \mid}$.

4.* Let (X,d) be a metric space. Show that finite union of bounded sets is again bounded.

5.* Show that for any $x \in \mathbb{R}^n$, $\|x\|_\infty \leq \|x\|_2 \leq \|x\|_1$. Also check that $\|x\|_1 \leq n\|x\|_\infty$ and $\|x\|_1 \leq \sqrt{n}\|x\|_2$.

6.* Show that $(l_\infty, \| \, \|_\infty)$ is a normed space. (These were defined in class).

7. Let (X,d) be a metric space. Show that every convergent sequence in X is Cauchy.

8. Two metrics d and ρ on a set M are said to be equivalent if they generate the same convergent sequences; that is, $d(x_n, x) \to 0$ if and only if $\rho(x_n, x) \to 0$. Show that on $\mathbb{R}^2$, the metrics given by the norms $\| \, \|_2$ and $\| \, \|_\infty$ are equivalent.

9.* Show that if (X,d) and (Y,ρ) are two metric spaces, $(X \times Y, \nu)$ is a metric space where $\nu((x_1,y_1),(x_2,y_2)) = d(x_1,x_2) + \rho(y_1,y_2)$. Further show that if $U \subset X$ and $V \subset Y$ are open then the set $U \times V$ is open in $X \times Y$.