Math 301 Tutorial 6

(1) State whether the following statements are true or false. If true, prove the statement. If false, explain why? No points will be awarded for only stating true or false.
  (a) Every sequence that has a convergent subsequence is bounded.
  (b) * If A is uncountable and B is countably infinite, then A and $A \cup B$ are equivalent.
  (c) Every Cauchy sequence has a convergent subsequence.
  (d) *Let (X, d) be a metric space and A be a subset of X. Then, for any $E \subset A$, $\text{int}_A(E) = A \cap \text{int}_X(E)$, where $\text{int}_A(E)$ denotes the interior of E in A and $\text{int}_X(E)$ denotes the interior of E in X.
  (e) Let (X, d) be a metric space. The function $\rho : X \times X \to \mathbb{R}$ defined as $\rho(x, y) = \dfrac{1}{1 + d(x, y)}$ is a metric on X.

(2) * Let (x_n) be a sequence in $\mathbb{R}$ where $x_1 > x_2 > 0$, such that (x_{2n}) is increasing and (x_{2n+1}) is decreasing. Given that $x_{n+1} = \dfrac{x_n + x_{n-1}}{2}$, show that (x_n) is convergent.

(3) * Let (X, d_1) and (Y, d_2) be metric spaces and $f : X \to Y$ and $g : X \to Y$ be such that $f(x) = g(x)$ on a dense set $D \subset X$. Show that $f(x) = g(x)$ for all $x \in X$.

(4) * Problem 1, page 64, Chapter 5 in Carothers.

(5) *Problem 32, page 67, Chapter 5 in Carothers.

(6) Problem 54, page 72 Chapter 5 Carothers.

(7) Let (X, d) be a metric space and D be a countable dense subset of X. Show that X cannot have uncountably many disjoint open subsets.

(8) (a) Show that $\| \|_\infty$ defines a norm on ℓ_2.

  (b) * Are the metrics induced by the norms $\| \|_\infty$ and $\| \|_2$ on ℓ_2 equivalent? Explain your answer.

(9) * Let F be the set of all $\underline{x} = (x_n)$ in ℓ_1 such that $x_n = 0$ for all but a finite number of $n \in \mathbb{N}$. Is F closed? or open? or neither?

(10) Let (X, d) be a metric space.

(a) Show that a convergent sequence (x_n) in X converges to a unique limit, that is, it cannot converge to more than one element in X.

(b) * Given that $\rho((x_1, x_2), (y_1, y_2)) = \sqrt{d(x_1, x_2)^2 + d(y_1, y_2)^2}$ is a metric on $X \times X$. Is the set $A = \{(x, x) \mid x \in X\} \subset X \times X$, closed in $(X \times X, \rho)$? Prove or disprove.