Math 301 Tutorial 8

(1) *

(a) Show that if (X, d) is complete then any decreasing sequence of closed sets
$$\cdots \subseteq F_n \subseteq \cdots F_2 \subseteq F_1$$
where diam $F_n \to 0$ has a non-empty intersection, that is, $\cap_n F_n \neq \emptyset$.

(b) Let (X, d) have the property that any decreasing sequence of closed sets
$$\cdots \subseteq F_n \subseteq \cdots F_2 \subseteq F_1$$
where diam $F_n \to 0$ has a non-empty intersection, that is, $\cap_n F_n \neq \emptyset$. Then show that every infinite totally bounded subset of X has a limit point in X.

(2) Let (X, d) be a metric space. Show that every Cauchy sequence with a convergent subsequence converges.

(3) * Show that (X, d) is compact if and only if for any decreasing sequence of of closed sets
$$A_1 \supseteq A_2 \supseteq \cdots A_n \supseteq \cdots$$
has a nonempty intersection, that is, $\cap_n A_n \neq \emptyset$.

(4) * Let (X, d) be a metric space, show that $(\ell_\infty(X), \| \ \|_\infty)$ is a complete metric space.

(5) * Show that every compact metric space (X, d) is separable.
(Hint: $X = \cup_x \mathrm{B}(x, \frac{1}{n})$ for all n.)

(6) Give an example of a bounded continuous map which is not uniformly continuous. Can an unbounded continuous function on $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous.

(7) Show that any function $f : \mathbb{R} \to \mathbb{R}$ is said to satisfy Lipschitz condition of order α, where $\alpha > 0$ is a real number, if there is a constant $K < \infty$ such that
$$|f(x) - f(y)| \leq K|x - y|^\alpha$$
for all $x, y \in \mathbb{R}$. Prove that such a function is uniformly continuous.

(8) * Fix $y \in \ell_\infty$ and define $g : \ell_1 \to \ell_1$ by $g((x_n)) = (x_n y_n)$. Show that g is uniformly continuous.