Theorem 0.1. *Let (X, d) be a complete metric space and $f : X \to X$ be a contraction mapping. Then f has a fixed point.*

Example 0.2. We would like to compute the cube root of 2. This is equivalent to solving the equation $x^3 - 2 = 0$.

Claim 0.3. There exists $\lambda \in \mathbb{R}$ such that for some choice of interval $[a, b] \subset \mathbb{R}$, the map $f : [a, b] \to [a, b]$ defined as $f(x) = x - \lambda(x^3 - 2)$ is a contraction mapping.

Proof. First note that we know that cube root of 2 lies in $[1, \frac{3}{2}]$ and therefore we can restrict our function to $[1, \frac{3}{2}]$.

We need to find a λ so that f is a contraction mapping.

$$\begin{aligned} |f(x) - f(y)| &= |x - y - \lambda(x^3 - y^3)| \\ &\leq |x - y||1 - \lambda(x^2 + y^2 + xy)| \end{aligned}$$

Now note that if we assume $x \in [1, \frac{3}{2}]$ then $(x^2 + y^2 + xy)$ has maximum value $\dfrac{27}{4}$ and minimum value 3, then if $\lambda < \dfrac{4}{27}$, we have $\lambda(x^2 + y^2 + xy) < 1$ and

$$1 - \lambda(x^2 + y^2 + xy) < 1.$$

Therefore, let $\lambda = \dfrac{1}{16}$. Now define,

$$f(x) = x - \frac{(x^3 - 2)}{16}.$$

Verify that $f(x)$ is a increasing function on $[1, \frac{3}{2}]$ and takes values in $[1, \frac{3}{2}]$.

Thus we have a function $f : [1, \frac{3}{2}] \to [1, \frac{3}{2}]$ which is a contraction mapping. Since $[1, \frac{3}{2}]$ is closed in $\mathbb{R}$, it is complete. Therefore, by previous theorem f has a fixed point.

Further, the fixed point of f is the cube root of 2. □

Now to find the approximate solution of cube root. We first start with a guess. For instance $x = 1.1$ may be close to the actual solution.

Take $x_1 = 1.1$ and compute

- $x_2 = f(x_1) = 1.1418125.$
- $x_3 = f(x_2) = 1.173377363488205$
- $x_4 = f(x_3) = 1.1977011220963$
- $x_5 = f(x_4) = 1.21532063080071$
- $x_6 = f(x_5) = 1.228131023304038.$

and so on, in this particular example we would need several more iterations to get correct approximation up to 2 decimals.