End Sem Exam: MTH-102A

LINEAR ALGEBRA AND ODE

Date: 23rd April, 2019 **Time: 4:00 PM - 7.00 PM** **Total Marks: 100**

Instructions:

1. *Please write down your name, roll no. and section no. in the answer book.*
2. *Please number the pages of the answer book. Make a tabular column on the top cover indicating the page number in which the respective question has been answered.*
3. *Answer all parts of a question together at one place.*

---

1. (a) Let $A = \begin{bmatrix} 3 & -6 & 0 \\ -6 & 0 & 6 \\ 0 & 6 & -3 \end{bmatrix}$. Find an orthogonal matrix U and a diagonal matrix D such that $A = UDU^T$. [10]

(b) Let $A = \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$ and let I be the 3×3 identity matrix. Find $a \in \mathbb{R}$ for which the nullity of $A - aI$ is maximum. [4]

(c) Let A, B, C are 2×2 diagonalizable matrices. The graphs of characteristic polynomials of A, B, C are shown below. From this information, determine the rank of the matrices A, B and C. [6]

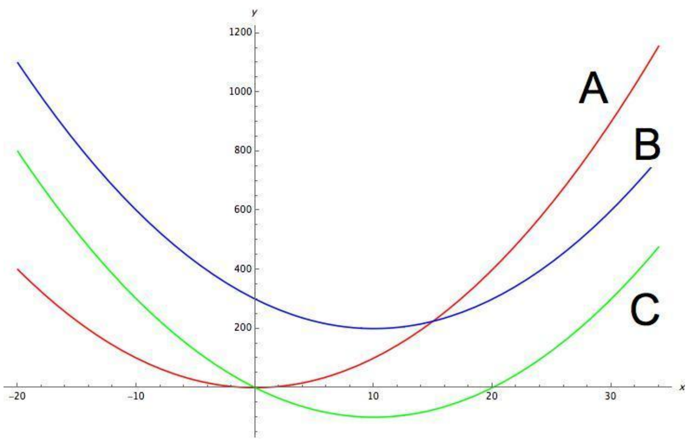

Figure 1: Graphs of characteristic polynomials

Marking Scheme:

The characteristic polynomial of A is

$$\begin{aligned}
p_A &= \begin{vmatrix} 3-\lambda & -6 & 0 \\ -6 & -\lambda & 6 \\ 0 & 6 & -3-\lambda \end{vmatrix} \\
&= (3-\lambda)\begin{vmatrix} -\lambda & 6 \\ 6 & -3-\lambda \end{vmatrix} - (-6)\begin{vmatrix} -6 & 6 \\ 0 & -3-\lambda \end{vmatrix} \\
&= (3-\lambda)(\lambda^2+3\lambda-36)+6(6\lambda+18) \\
&= -\lambda^3+81\lambda \\
&= -\lambda(\lambda-9)(\lambda+9)
\end{aligned}$$

[2]

Hence, the eigenvalues are $0, 9$, and -9.

The RREF of A is $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$. Hence, a basis for $N(A)$ is given by the single vector $v_1 = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 1 \end{bmatrix}$.

The RREF of $A-9I$ is $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$. Hence, a basis for $N(A-9I)$ is given by the single vector $v_2 = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

The RREF of $A-(-9)I$ is $\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Hence, a basis for $N(A-(-9)I)$ is given by the single vector $v_3 = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix}$ [4]

The vectors $\{v_1, v_2, v_3\}$ is an orthogonal set of eigen-vectors for the matrix A. [1]

Normalizing we get $\{u_1 = \frac{1}{\|v_1\|}v_1 = \frac{2}{3}v_1 = \begin{bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, u_2 = \frac{1}{\|v_2\|}v_2 = \frac{1}{3}v_2 = \begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, u_3 = \frac{1}{\|v_3\|}v_3 = \frac{2}{3}v_3 = \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ \frac{2}{3} \end{bmatrix}\}$ is a orthonormal set of eigen vectors. [2]

Let $U = \frac{1}{3}\begin{bmatrix} 2 & -2 & -1 \\ 1 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -9 \end{bmatrix}$. Then $A = UDU^T$. [1]

Marking Scheme:

Note that if a is not an eigenvalue of the matrix A, then the matrix $A-aI$ is invertible, that is, the null space of $A-aI$ is the zero space, and the dimension is 0. So a should be an eigenvalue.

[1]

The characteristic polynomial of A is $t(6-t)^2$. 0 and 6 are the eigen values with multiplicity 1 and 2 respectively. [1]

Since the matrix A is a real symmetric matrix, it is diagonalizable and hence the null space of $A - 6I$ has exactly two linearly independent eigen vectors, whereas the null space of $A - 0I$ is of dimension 1. So $a = 6$. [2]

Note: If you are doing it in other methods then you should explain why 6 is the only value possible and others are ruled out. Without this explanation you may lose 2 marks.

Marking Scheme:

The graph of the characteristic polynomial $p_A(\lambda)$ of A passes through origin. So $\lambda = 0$ is an eigen value of A and since the x-axis is tangential to the graph of $p_A(\lambda)$, 0 is the only root of $p_A(\lambda)$ with multiplicity 2. Since A is diagonalizable it is similar to the zero matrix and hence $rank(A) = 0$. [2]

The graph of characteristic polynomial $p_B(\lambda)$ of B does not pass through the origin. Thus $\lambda = 0$ is not an eigenvalue of B. Since B is diagonalizable, it is similar to a diagonal matrix with diagonal entries nonzero. So $rank(B) = 2$. [2]

For the matrix C, 0 is a simple root of the characteristic polynomial. So C is similar to a diagonal matrix with exactly one of the diagonal entry is nonzero. So $rank(C) = 1$. [2]

2. (a) Does the function $f(x,y) = \sqrt{y} + 1$ satisfy Lipschitz condition in any rectangle containing the origin ? What can you say about the existence and uniqueness of solution of the IVP $y' = \sqrt{y} + 1, \;\; y(0) = 0, \;\; x \in [0,1]$? Give proper justification to your answer.

[4+4]

Marking Scheme:

Consider any rectangle $R = [0, a] \times [0, b]$ containing origin. We have

$$\frac{|f(x,y_1) - f(x,y_2)|}{|y_1 - y_2|} = \frac{|\sqrt{y_1} - \sqrt{y_2}|}{|y_1 - y_2|} = 1/\sqrt{\delta}, \quad \text{for } y_1 = \delta > 0, \quad y_2 = 0.$$

[2]

For δ arbitrary small, we can make $\frac{|f(x,y_1)-f(x,y_2)|}{|y_1-y_2|}$ arbitrarily large on R. Hence f does not satisfy Lipschitz condition in any rectangle containing origin. [2]

Since $f(x,y)$ is continuous at $(0,0)$, the IVP has a solution by Picard theorem. [1]

Let $g_1(x), \; g_2(x)$ be two solutions of the IVP. Consider $z(x) = (\sqrt{g_1} - \sqrt{g_2})^2$. Then $z'(x) = -\frac{z(x)}{\sqrt{g_1}\sqrt{g_2}} \leq 0$. [1]

Thus $z(x)$ is a decreasing function. Further $z(x)$ is non negative and $z(0) = 0$. Then $z(x) = 0$ for all $x \geq 0$. Hence $g_1 = g_2$.

[2]

Remark: If you have solved the equation explicitly, then [4] marks are awarded for the last [4] marks breakup.

(b) Let $\Omega = \mathbb{R}^2 - (0,0)$. Consider the functions $M, N : \Omega \to \mathbb{R}$ defined by $M(x,y) = \frac{-y}{x^2+y^2}$ and $N(x,y) = \frac{x}{x^2+y^2}$. Show that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ on Ω. Is the differential $Mdx + Ndy$ exact i.e. does there exist a differentiable function $u : \Omega \to \mathbb{R}$ such that $M = \frac{\partial u}{\partial x}$ and $N = \frac{\partial u}{\partial y}$? Justify your answer. [2+6]

Marking Scheme:

- $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$. [2]
- If such a u exists then $Mdx + Ndy = du$ and so by Fundamental theorem of Line integral $\oint_C Mdx + Ndy = 0$ for ANY closed curve C in Ω (C need NOT be SIMPLE). [2]
- But the line integral $\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = 2\pi$ where C is the unit circle with anticlockwise orientation.

  To see that, consider $x = \cos t, y = \sin t, \;\; 0 \leq t \leq 2\pi$. Then

  $$\oint_C \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = \int_0^{2\pi} (\cos^2 t + \sin^2 t)dt = 2\pi.$$

  So the differential is not exact.

  (**Remark:** If you have written the value of the integral as 2π but have NOT shown calculation, then 0 mark is awarded.) [4]

- **Explanation why the choice $u(x, y) = \tan^{-1}(y/x)$ does not work:**

  Fist of all, for a given $(x, y) \neq (0, 0)$, $\tan^{-1}(y/x)$ can have many values. Even if we fix a range length of 2π, for example $[0, 2\pi)$, then $\tan^{-1}(y/x)$ becomes a well defined function on $\mathbb{R}^2 - (0, 0)$. But still it does not satisfy the required condition. We can see it as follows: Observe that any u which satisfies $\nabla u(x, y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$ for all $(x, y) \neq (0, 0)$ has to be differentiable on $\mathbb{R}^2 - (0, 0)$ (since the partial derivatives are continuous.) But our choice of $\tan^{-1}(y/x)$ is discontinuous along positive x-axis (since it takes value small positive values just above positive x-axis and takes value near 2π just below the positive x-axis.)

(c) Solve the first order equation: $y' = y(xy^3 - 1)$. [4]

Marking Scheme:

[Recall that Bernoulli equation is of the form $y' + P(x)y = Q(x)y^n$. To solve it, we have to change variable to $z = y^{1-n}$. Then it reduces to linear ODE $z' + (1 - n)P(x)z = (1 - n)Q(x)$.]

The ODE is $y' + y = xy^4$. Substitute $u = 1/y^3$.

[1]

We get $u' - 3u = -3x$. [1]

Using integrating factor e^{-3x}, we write

$$\frac{d}{dx}(ue^{-3x}) = -3xe^{-3x} \implies ue^{-3x} = \frac{1 + 3x}{3}e^{-3x} + C \implies u = \frac{1 + 3x}{3} + Ce^{3x}.$$

Hence, the solution is $\frac{1}{y^3} = Ce^{3x} + x + 1/3$. [2]

3. (a) The normal form of the Bessel equation is

$$y'' + (1 + \frac{1/4 - p^2}{x^2})y = 0, \quad x > 0.$$

Let y_p be a non-trivial solution of it on the positive x-axis. If $0 \leq p < 1/2$, then show that for every interval $[a, a+\pi]$, $a > 0$, of length π there is a zero of y_p in $(a, a+\pi)$. If $x_1 < x_2$ be consecutive positive zeros of y_p, then show that $x_2 - x_1$ is less than π and $x_2 - x_1$ approaches π as $x_1 \to \infty$.

[10]

Marking Scheme: Given equation $y'' + q(x)y = 0, \quad x > 0 - - - - - - - - - - - -(1)$

where $q(x) = 1 + \frac{1/4-p^2}{x^2} > 1$. [1]

Take $v'' + v = 0 - - - - - - - - -(2)$. [1]

By Sturm comparison theorem between two consecutive roots of $v(x)$ there is a root of y_p. Consider an interval $[a, a+\pi]$. Take $v(x) = \sin(x-a)$ a solution of (2). Then $a, a+\pi$ are two consecutive roots of $v(x)$. By SCT y_p must vanish in between two consecutive zeros $a, a+\pi$ of $v(x)$. Hence there exists a zero of y_p in $(a, a+\pi)$. [2]

(**Remark:** If you have used $v(x) = \sin x$ in this step, then the conclusion can not be reached using Sturm comparison theorem and hence 0 mark is given in that case.)

Let x_1, x_2 be two consecutive zeros of y_p. Take the interval $[x_1, x_1+\pi]$. By above $x_2 < x_1 + \pi$. Hence $x_2 - x_1 < \pi$. [1]

For any given $\epsilon > 0$, we can find x' such that $1 < q(x) < 1 + \epsilon$ for all $x > x'$. [1]

Here take $x_1 > x'$. [1]

Consider the equation $w'' + (1+\epsilon)w = 0$. Take solution $w = \sin[(x - x_1)\sqrt{1+\epsilon}]$. [1]

Then $w(x_1) = 0$ and next zero is $x_1 + \frac{\pi}{\sqrt{1+\epsilon}}$. By SCT we must have $x_1 + \frac{\pi}{\sqrt{1+\epsilon}} < x_2$ i.e. $x_2 - x_1 > \frac{\pi}{\sqrt{1+\epsilon}}$. As $\epsilon \to 0$, we have $x' \to \infty$ and so $x_1 \to \infty$. [2]

Remark: If you have argued that $q(x) \to 1$ as $x \to \infty$ and the roots of $v'' + v = 0$ are π apart. Hence $x_2 - x_1 \to \pi$. This is an imprecise but intuitively right answer. In this case [3] marks are awarded in place of last [5] marks breakup.

(b) Solve the IVP by Laplace transforms:

$$y''(t) + 9y(t) = \begin{cases} 8\sin t & \text{if } 0 < t < \pi \\ 0 & \text{if } t > \pi \end{cases} \qquad y(0) = 0, y'(0) = 4.$$ [6]

Marking Scheme:

Let $r(t) = 8(u(t) - u(t-\pi))\sin t = 8u(t)\sin t + u(t-\pi)\sin(t-\pi)$. [1]

Taking Laplace Transform on both sides of the ODE, we get

$$(s^2+9)Y(s) = R(s) + 4 \implies Y(s) = \frac{4}{s^2+9} + \frac{R(s)}{s^2+9}$$

[2]

We can explicitly write $R(s)$ and then use partial fraction technique.

$$Y(s) = \frac{4}{s^2+9} + (1+e^{-\pi s})\frac{8}{(s^2+1)(s^2+9)} = \frac{4}{s^2+9} + (1+e^{-\pi s})\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right)$$

[2]

This gives

$$\begin{aligned} y(t) &= \frac{4}{3}\sin 3t + \left(\sin t - \frac{1}{3}\sin 3t\right) + u(t-\pi)\left(\sin(t-\pi) - \frac{1}{3}\sin 3(t-\pi)\right) \\ &= \sin t + \sin 3t + u(t-\pi)\left(\frac{1}{3}\sin 3t - \sin t\right) \end{aligned}$$

[1]

(

Otherwise, use convolution as follows

$$y(t) = \frac{4}{3}\sin 3t + \frac{1}{3}\int_0^t r(\tau)\sin 3(t-\tau)\,d\tau$$

Thus for $0 < t < \pi$, we get

$$y(t) = \frac{4}{3}\sin 3t + \frac{8}{3}\int_0^t \sin\tau \sin 3(t-\tau)\,d\tau = \frac{4}{3}\sin 3t + \sin t - \frac{1}{3}\sin 3t = \sin 3t + \sin t$$

and for $t > \pi$, we get [since $r(t) = 0$]

$$y(t) = \frac{4}{3}\sin 3t + \frac{8}{3}\int_0^\pi \sin\tau \sin 3(t-\tau)\,d\tau + \frac{1}{3}\int_\pi^t 0\ \sin 3(t-\tau)\,d\tau = \frac{4}{3}\sin 3t$$

This solution matches with that obtained earlier.)

(c) Solve the Integral equation by Laplace transform: $e^{-t} = y(t) + 2\int_0^t \cos(t-\tau)y(\tau)d\tau$.

[4]

Marking Scheme:

Taking Laplace Transform, we get

$$Y(s) = \frac{s^2+1}{(s+1)^3} = \frac{1}{1+s} - \frac{2}{(s+1)^2} + \frac{2}{(s+1)^3}$$

[2]

Thus,

$$y(t) = e^{-t}(t-1)^2$$

[2]

4. (a) (i) Prove that the zeros of an analytic function $f(x)$, which is not identically zero, are isolated points i.e. if $f(x_0) = 0$ then there exists $\epsilon > 0$ such that $f(x) \neq 0$ for all $0 < |x - x_0| < \epsilon$.

(ii) Deduce that if f, g are analytic functions on an interval I and the Wronskian $W(f, g) = 0$ on I then f, g are linearly dependent on I.

[4+4]

Marking Scheme:

(i) Write $f(x) = \sum_{n \geq 0} a_n (x - x_0)^n$ on $|x - x_0| < R$ for some $R > 0$. Since a power series can be differentiated term by term, we get $n! a_n = f^{(n)}(x_0)$. Since $f(x_0) = 0$, we have $a_0 = 0$. Since f is not zero function there exists m such that $a_m \neq 0$.

[1]

Choose m to be the least such that $a_m \neq 0$. Then $f(x) = a_m(x - x_0)^m + a_{m+1}(x - x_0)^{m+1} + \cdots = (x - x_0)^m [a_m + a_{m+1}(x - x_0) + \cdots] = (x - x_0)^m g(x)$ where g is analytic and $g(x_0) = a_m \neq 0$.

[1]

By continuity of g, there exists exists $\epsilon > 0$ such that $g(x) \neq 0$ for all $|x - x_0| < \epsilon$. Hence $f(x) \neq 0$ for all $0 < |x - x_0| < \epsilon$. [2]

(ii) Given that $fg' - f'g = 0$ on an interval I. Since zeros of f are isolated points we can choose an interval $I' \subset I$ such that $f \neq 0$ on I'.

[1]

Then on I', we have $(fg' - f'g)/f^2 = 0$, implies $(g/f)' = 0$, imples $g = cf$ on I'.

[2]

Now $h = g - cf$ is analytic on I and h is zero on an interval I' i.e. h has non isolated zero. Hence by (i), we must have $h = 0$ on I. [1]

Remark: If you have written: $fg' - gf' = 0 \implies f'/f = g'/g \implies f = cg$ by integrating. This is a wrong argument, since it shows that if just Wronksian is zero then the functions are dependent. This is not true as we have done the example in class $x^2, x|x|$. So in this case 0 mark is given.

(b) Consider the equation $y'' - 2xy' + 2my = 0$ where m is a positive integer. Show that $x = 0$ is an ordinary point and one of the power series solutions about $x = 0$ is a polynomial of degree m. [8]

Marking Scheme:

Comparing with $y'' + p(x)y' + q(x)y = 0$ we have $p(x) = -2x, q(x) = 2m$. Both of them are analytic at the origin. So $x = 0$ is an ordinary point. [1]

Assume solution $y = \sum a_n x_n$. Then $y' = \sum n a_n x^{n-1}$ and $y'' = \sum n(n-1) a_n x^{n-2}$. [1]

Substituting in the given equation: $\sum n(n-1) a_n x^{n-2} - 2x \sum n a_n x^{n-1} + 2m \sum a_n x_n = 0$.

Coff of x^n is $(n+2)(n+1)a_{n+2} - 2na_n + 2ma_n$. Equating it to 0, we get

$$a_{n+2} = 2\frac{n-m}{(n+2)(n+1)}a_n.$$

[2].

Then $a_{m+2} = a_{m+4} = \cdots = 0$.

General solution: $y = a_0(1 - mx^2 - \frac{m(2-m)}{6}x^4 \cdots) + a_1(x - \frac{(1-m)}{3}x^3 + \cdots) = a_0y_1 + a_1y_1$.

[2]

So if m is odd y_2 becomes a polynomial of degree m. If m is even, then y_1 becomes a polynomial of degree m.

[2]

(c) Solve $x^2y'' + 2xy' - 12y = 0$. [4]

[Recall: The ODE of the form $x^2\frac{d^2y}{dx^2} + ax\frac{dy}{dx} + by = 0$, where a, b are constants, is called the Cauchy-Euler equation. Under the transformation $x = e^t$ (when $x > 0$) for the independent variable, the above reduces to $\frac{d^2y}{dt^2} + (a-1)\frac{dy}{dt} + by = 0$, which is an equation with constant coefficients.]

Marking Scheme:

Using the substitution $x = e^t$, the given equation reduces to

$$\frac{d^2u}{dt^2} + \frac{du}{dt} - 12u = 0$$

[2]

$$\implies m^2 + m - 12 = 0 \implies m = -4, 3 \implies u(t) = Ae^{-4t} + Be^{3t} = y(e^t).$$

[1]

The general solution is thus

$$y(x) = \frac{A}{x^4} + Bx^3.$$

[1]

5. (a) Find a particular solution by *operator method* and find the general solution of the ODE:

$$y''' - 3y'' - y' + 3y = x^2 e^x.$$

[6]

Marking Scheme:

Characteristic equation $m^3 - 3m^2 - m + 3 = 0 \implies m = -1, 1, 3.$ [1]

Hence homogeneous solution $y_h = Ae^{-x} + Be^x + Ce^{3x}$. [1]

Now $r(x) = x^2 e^x$. Let $D \equiv d/dx$ and y_p be the particular solution. Then

$$\frac{1}{D^3 - 3D^2 - D + 3} x^2 e^x = \frac{1}{(D-1)^3 - 4(D-1)} x^2 e^x = e^x \frac{1}{D^3 - 4D} x^2$$

[2]

$$= e^x \frac{1}{-4D(1 - D^2/4)} x^2 = e^x \frac{1}{-4D}(1 + D^2/4 + \cdots)x^2 = e^x \frac{1}{-4D}(x^2 + \frac{1}{2}) = -\frac{e^x}{4}(\frac{x^3}{3} + \frac{x}{2}).$$

So the particular integral is

$$y_p(x) = -e^x \left(\frac{x}{8} + \frac{x^3}{12} \right).$$

[2]

Thus the general solution is

$$y = Ae^{-x} + Be^x + Ce^{3x} - e^x \left(\frac{x}{8} + \frac{x^3}{12} \right).$$

Remark: If you have not calculated particular integral by operator method, then 0 mark.

(b) Given that $y = x$ is a solution of the homogeneous part of the following ODE, find its general solution.

$$x^2 y'' - x(x+2)y' + (x+2)y = x^3, \quad x > 0.$$

[8]

Marking Scheme: $y_1 = x$ is a solution of the homogeneous part. To find another linearly independent solution we assume $y = xu$. This gives

$$u'' - u' = 0 \implies u' - u = 1 \implies u = e^x - 1 \implies y = xe^x - x$$

[3]

Since $y_1 = x$, we take $y_2 = xe^x$. The nonhomogeneous part is written as

$$y'' - \frac{x+2}{x} y' + \frac{(x+2)}{x^2} y = x.$$

Thus $r(x) = x$ and $W(y_1, y_2) = x^2 e^x$. [1]

Now

$$u = -\int \frac{y_2 r}{W} dx = -x$$

and

$$v = \int \frac{y_1 r}{W} dx = -e^{-x}$$

Thus $y_p = -x - x^2$. [3]

General solution: (absorbing first term of y_p in the homogeneous solution)

$$y = x(A + Be^x) - x^2.$$

[1]

(c) Consider an ODE of the form $y'' + p(x)y + q(x)y = 0$.

(i) Show that if $p(x)$ and $q(x)$ are continuous functions for all x, a solution whose graph is tangent to the x-axis at some point must be identically zero.

(ii) Find an equation of the above form having x^2 as a solution by calculating its derivatives and finding a linear equation connecting them. Why isnt part (i) contradicted, although the graph of x^2 is tangent to the x axis at 0 ? [3+3]

Marking Scheme:

(i) Let $y(x)$ be a solution whose graph is tangent to the x-axis at $x = a$. Then $y(a) = y'(a) = 0$.

[1]

But $z(x) = 0$ is also a solution of the given ODE with same initial value $z(a) = z'(a) = 0$.

[1]

Hence by Existence and Uniqueness of solution of second order liner ODE we must have $y = z$ i.e. $y = 0$.

[1]

(ii) Let $y = x^2$. Then $y' = 2x, y'' = 2$. We have $xy'' = 2x = y'$. So we get $y'' - y'/x = 0$.

[1]

This is not a contradiction to part (i) since here $p(x) = 1/x, q(x) = 0$. So $p(x)$ is not continuous at the origin. So we can not apply Existence and Uniqueness Theorem. Hence there is no contradiction.

[2]

Remark: Many of you have argued that: Putting $y = x^2$ in the above equation $2 + p(x)2x + q(x)x^2 = 0 \implies 2/x = -2p(x) + q(x)x$ for $x \neq 0$. The RHS is continuous at 0 but LHS is not.

This argument shows x^2 can not be a solution of differential equation of the above form with p, q continuous at the origin.

BUT this is not what the question asked and hence 0 mark given. According to the question:

step1: You have to find an ODE of the above form which has x^2 as solution ($y'' = 2$ is NOT in the above form since it is not homogeneous).

step 2: Then you have to explain why the this is not a contradiction to conclusion of part (i).