MID-SEMESTER EXAMINATION
SPRING-2019
LINEAR ALGEBRA AND ODE, MTH-102A

Time Allowed: 2 hrs Max. Marks: 60

(1) (a) Solve the following system of linear equations by using the Gauss-Jordan elimination (row reduced echelon form) method. [6]

$$x + y + z = 5$$
$$2x + 3y + 5z = 8$$
$$4x + 5z = 2$$

Solution: The RREF of the augmented matrix is $[I|(3,4,-2)^T]$, where I is the 3×3 identity matrix. [5]

So the solution is $x = 3, y = 4, z = -2$. [1]

Mark Distribution: Even if the answer is correct, if someone you haven't got the RREF of the augmented matrix as above then you will be awarded 2.

(b) Let A be an $n \times n$ real matrix with $a_{ij} = \max\{i, j\}$ for $i, j = 1, 2, \ldots, n$. Calculate the determinant of A. [6]

Solution: If we replace R_1 by $R_1 - R_2$ and then R_2 by $R_2 - R_3$ and so on R_{n-1} by $R_{n-1} - R_n$ then we will get a lower triangular matrix with all diagonal entries being -1 except $a_{nn} = n$, so $\det A = n(-1)^{n-1}$.

[6]

(Full marks for correct answer with proper justification by any method otherwise 0.)

(2) (a) Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Let W_1 be the subset of even functions, $f(-x) = f(x)$ and let W_2 be the subset of odd functions, $f(-x) = -f(x)$. Show that [6]
(i) W_1 and W_2 are subspaces of V.
(ii) $V = W_1 + W_2$ and $W_1 \cap W_2 = \{0\}$.

SOLUTION:

(i) Let $f_1, f_2 \in W_1$. Then $(cf_1 + f_2)(-x) = cf_1(-x) + f_2(-x) = cf_1(x) + f_2(x) = (cf_1 + f_2)(x)$. So $cf_1 + f_2$ is an even function and hence belongs to W_1. So W_1 is a subspace of V. [1]

Similarly let $g_1, g_2 \in W_2$. Then $(cg_1 + g_2)(-x) = cg_1(-x) + g_2(-x) = -cg_1(x) - g_2(x) = -(cg_1 + g_2)(x)$. So $cg_1 + g_2$ is an odd function and hence belongs to W_2. So W_2 is a subspace of V. [1]

(ii) Let $f \in V$ then $f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x))$. [2]

The function $g(x) = \frac{1}{2}(f(x) + f(-x))$ is an even function and the function $h(x) = \frac{1}{2}(f(x) - f(-x))$ is an odd function and $f(x) = g(x) + h(x)$ where $g(x) \in W_1$ and $h(x) \in W_2$ and hence $V = W_1 + W_2$. [1]

For $W_1 \cap W_2 = \{0\}$ we need to show that a function is both odd and even implies it is a zero function. Let $f(x) \in W_1 \cap W_2$. Then $f(-x) = -f(x) = f(x)$. So $f(x) = 0$ for all x. Hence $f \equiv 0$. [1]

(b) Let $\mathcal{P}_{2n}(x)$ be the vector space of all polynomials in x of degree at most $2n$ with real coefficients. Let $W = \{P \in \mathcal{P}_{2n}(x) : P \text{ has only terms of even degree and } P(1) + P(-1) = 0\}$. Find a basis for W and compute $dim(W)$. [6]

Solution: Let $P(x) = a_0 + a_2x^2 + \cdots + a_{2n}x^{2n}$. Then $P(1) + P(-1) = 0$ implies $a_0 + a_2 + \cdots + a_{2n} = 0$. So $a_0 = -(a_2 + \cdots + a_{2n})$. [1]

So $P(x) = a_2(x^2 - 1) + a_4(x^4 - 1) + \cdots + a_{2n}(x^{2n} - 1)$. So the set $\{x^2 - 1, x^4 - 1, \cdots, x^{2n} - 1\}$ spans the space. [2]

You need to show that the set $\{x^2 - 1, x^4 - 1, \cdots, x^{2n} - 1\}$ is LI. [2]

So $\{x^2 - 1, x^4 - 1, \cdots, x^{2n} - 1\}$ is a basis of W and so $dim(W) = n$. [1]

(3) (a) Let $M_2(\mathbb{R})$ be the vector space of all 2×2 matrices with real entries and let $\mathcal{P}_3(x)$ be the vector space of real polynomials in x of degree at most 3. Define the linear map $T : M_2(\mathbb{R}) \to \mathcal{P}_3(x)$ by $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = 2a + (b-d)x - (a+c)x^2 + (a+b-c-d)x^3$. Find the Rank and Nullity of T and verify the Rank-Nullity theorem for T. [8]

Solution: The set $B_1 = \{\mathbf{e}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \mathbf{e}_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$ is a basis for $M_2(\mathbb{R})$ and $B_2 = \{1, x, x^2, x^3\}$ is a basis for $P_3(x)$. [1]

Now $T(\mathbf{e}_1) = 2 - x^2 + x^3$, $T(\mathbf{e}_2) = x + x^3$, $T(\mathbf{e}_3) = -x^2 - x^3$ and $T(\mathbf{e}_4) = -x - x^3$. [2]
So the matrix of T with respect to the bases B_1 and B_2 is

$$[T]_{B_1}^{B_2} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 \\ 1 & 1 & -1 & -1 \end{bmatrix}.$$

The rref of $[T]_{B_1}^{B_2}$ is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So $rank(T) = 3$. (for finding rank correctly by any method) [3]

$Nullity(T)$=Number of non-pivot columns in $[T]_{B_1}^{B_2} = 1$. (This can also be calculated directly). [1]

So $Rank(T) + Nullity(T) = 3 + 1 = 4 = dim(M_2(\mathbb{R}))$. [1]

(b) Construct a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $Ker(T) = Image(T)$. Is it possible to construct a linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$ having the same property ? Justify your answer. [4]

Solution: For $\mathbb{R}^2 \to \mathbb{R}^2$, we can consider the following linear map: $(x, y) \mapsto (y, 0)$. Then the image is equal to the kernel. [2]
(Other examples are $T(x, y) = (x - y, x - y),\ T(x, y) = (0, x)$.)

In general, the rank-nullity theorem tells us that the sum of the dimensions of the kernel and image is equal to the dimension of the domain of a linear transformation. In particular, there's no linear transformation $\mathbb{R}^3 \to \mathbb{R}^3$ which has the same dimensions of the image and kernel, because 3 is odd. [2]

(4) (a) Find the values of k for which the matrix $A = \begin{bmatrix} 2 & -2 & k \\ -1 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix}$ is similar to a diagonal matrix over $\mathbb{R}$. [7]

Solution: We use the fact that A is diagonalisable iff the minimal polynomial splits as distinct linear factors over $\mathbb{R}$. [1]

The characteristic polynomial of the matrix A is $(2-\lambda)(\lambda^2 - 4\lambda + 2k + 2)$. [1]

Let $P(\lambda) = \lambda^2 - 4\lambda + 2k + 2$. It's discriminant is $8 - 8k$.

If $k = 1$ then the only eigenvalue is 2. Since $A \neq 2I$, the minimal polynomial is NOT $(x-2)$. So it is not similar to a diagonal matrix. [2]

If $k > 1$ then the discriminant is negative so $P(\lambda)$ has complex roots. So A is not diagonalizable over $\mathbb{R}$. [1]

If $k < 1$ then the discriminant is positive and so $P(\lambda)$ has two distinct REAL roots other than 2. So the characteristic polynomial is a product of distinct linear factors and hence A is diagonalizable OVER $\mathbb{R}$. [2]

(Note that if A is diagonalisable then it may not have distinct eigen values. So those of you said that A is diagonalizable then it MUST have distinct eigen values is NOT correct)

(b) Let A be a 10×10 matrix with real entries such that $A^{2019} = 0$. Prove that $A^{10} = 0$. [5]

Solution: Let λ be an eigen value of A. Then there exists a non-zero column vector X such that $AX = \lambda X$. Then $A^{2019}X = \lambda^{2019}X$. So $\lambda^{2019}X = 0$ and since $X \neq 0$ we have $\lambda = 0$. So all the eigen values of A are 0. [3]

So the characteristic polynomial of A is x^{10}. Hence by Cayley-Hamilton theorem $A^{10} = 0$. [2]

Alternate Solution: Let $g(x) = x^{2019}$. Since $g(A) = 0$, the minimal polynomial say $m(x)$ divides $g(x)$. [2]

So $m(x) = x^k$ for some $k \leq 10$. [2]

Hence $A^k = 0$, $k \leq 10$ and thus $A^{10} = 0$. [1]

(5) (a) Let V be the vector space of all continuous functions from $[0, \pi]$ to $\mathbb{R}$ with the inner product

$$(f, g) = \int_0^\pi f(t)g(t)dt.$$

Let $W = Span(S)$, where $S = \{1, t, sin(t), cos(t)\}$. Use Gram-Schmidt orthogonalization process to S to obtain an orthonormal basis for W. [7]

Solution: Let $v_1 = 1, v_2 = t, v_3 = sin(t)$ and $v_4 = cos(t)$.
$w_1 = v_1 = 1$.
$w_2 = v_2 - \frac{(v_2,w_1)}{\|w_1\|^2} w_1 = t - \frac{\pi}{2}$. [1]

$w_3 = v_3 - \frac{(v_3,w_1)}{\|w_1\|^2} w_1 - \frac{(v_3,w_2)}{\|w_2\|^2} w_2$. Note that $(v_3, w_2) = 0$. So $w_3 = sin(t) - \frac{2}{\pi}$. [2]

$w_4 = v_4 - \frac{(v_4,w_1)}{\|w_1\|^2} w_1 - \frac{(v_4,w_2)}{\|w_2\|^2} w_2 - \frac{(v_4,w_3)}{\|w_3\|^2} w_3$. Note that $(v_4, v_1) = (v_4, v_3) = 0$. So $w_4 = cos(t) + \frac{24}{\pi^3}(t - \frac{\pi}{2})$. [2]

$\|1\| = \sqrt{\pi}$, $\|t - \frac{\pi}{2}\| = \sqrt{\frac{\pi^3}{12}}$, $\|sint - \frac{2}{\pi}\| = \sqrt{(\frac{\pi}{2} - \frac{4}{\pi})}$.
Final answer after normalization. [2]

(b) Let $M_3(\mathbb{R})$ be the vector space of all 3×3 real matrices with the inner product $(A, B) = Tr(A^T B)$. Find the orthogonal complement of the subspace of diagonal matrices. [5]

Solution: Let W be the subspace of diagonal matrices. By definiton $W^\perp = \{A \in M_3(\mathbb{R}) : Tr(A^T B) = 0 \ \forall \ B \in W\}$. Let $A = (a_{i,j})$ and $B = diag(d_1, d_2, d_3)$. Then $Tr(A^T B) = 0$ implies $a_{11}d_1 + a_{22}d_2 + a_{33}d_3 = 0$. Since d_1, d_2 and d_3 are arbitrary by taking $d_1 = 1, d_2 = 0$ and $d_3 = 0$ we get $a_{11} = 0$. Similarly by taking $d_1 = 0, d_2 = 1$ and $d_3 = 0$ we get $a_{22} = 0$ and by taking $d_1 = 0, d_2 = 0$ and $d_3 = 1$ we get $a_{33} = 0$. [3]

On the other hand if A is a 3×3 matrix with zeros on the main diagonal then $Tr(A^T B) = 0$ for all diagonal matrix B. So the orthogonal complement is the set of all 3×3 matrices with zeros on the main diagonal. [2]