QB: B

Name: Roll No: Section:

Quiz-II, MTH-102A: LINEAR ALGEBRA AND ODE

Date: 5th April, 2019 **Time: 6.30 PM-7.00 PM** **Max. Marks: 20**

1. **Write your Name, Roll No. and Section.**
2. **Write answers only in the space provided.**
3. **No additional sheets will be provided.**

1. Find a suitable integrating factor and solve the equation.

$$xy' + (3 - 4x)y = \frac{e^{4x}}{x}, \quad x > 0.$$

[3]

Answer:

Solution: For $y' + p(x)y = r(x)$, a integrating factor is $\mu(x) = e^{\int p dx}$. Here

$$p(x) = \frac{3}{x} - 4, \quad r(x) = \frac{e^{4x}}{x^2}.$$

So

$$\mu(x) = x^3 e^{-4x}.$$

[2]

Hence the solution is $y\mu(x) = \int r(x)\mu(x) = x^2/2 + c$. So

$$y = c\frac{e^{4x}}{x^3} + \frac{e^{4x}}{2x}.$$

[1]

2. Which pairs of functions below can be an independent set of solutions for $y''+p(x)y'+q(x)y=0$ with $p(x), q(x)$ being continuous on the interval as indicated below? Justify your answer for each case.

   (a) $y_1 = x^3, \quad y_2 = x^2, \quad x \in (0, \infty)$.

   (b) $y_1 = x, \quad y_2 = \sin x, \quad x \in (0, \infty)$.

   (c) $y_1 = 0, \quad y_2 = e^{4x}, \quad x \in (-\infty, \infty)$.

[5]

Solution: Clearly functions in (c) are linearly dependent. [1]

For (b) the wronskian is $(x\cos x - \sin x)$. It has zero at infinitely many points on $(0, \infty)$. So they can not be independent solution of second order linear homogeneous ODE. [2]

For (a) the wronskian is $(2x^4 - 3x^4 = -x^4)$. It is NEVER zero on $(0, \infty)$. So they are independent solution of second order linear homogeneous ODE. [2]

3. Find the general solution of the following equation using variation of parameters.

$$y'' + 2y' + y = \frac{e^{-x}}{x}, \quad x > 0.$$

[6]

Solution:

- Characteristic equation of the homogeneous part : $m^2 + 2m + 1 = 0$, so $m = -1, -1$. [1]
- Two independent solution of the homogeneous part $y_1 = e^{-x}, \quad y_2 = xe^{-x}$. [1]
- Wronskian $W(y_1, y_2) = e^{-2x}$. [1]

  Particular integral $y_p = u(x)y_1 + v(x)y_2$.
- $u(x)' = -\frac{y_2 r}{W} = -1$ so $u(x) = -x$ [1]
- $v(x)' = \frac{y_1 r}{W} = 1/x$ so $v(x) = \ln(x)$. [1]
- So particular integral $y_p = -xe^{-x} + \ln(x)xe^{-x}$.
- General solution:$y = c_1 e^{-2x} + c_2 xe^{-x} - xe^{-x} + \ln(x)xe^{-x}$.

  or $y = c_1 e^{-x} + c_2 xe^{-x} + \ln(x)xe^{-x}$ [1]

4. Consider the IVP.

$$(x^2 - 2x)y' = 2(x-1)y, \qquad y(x_0) = y_0.$$

(a) For which values of (x_0, y_0), Picard's theorem implies unique solution of the the IVP?

(b) Determine all values of (x_0, y_0) such that the IVP has no solution.

(c) Determine all values of (x_0, y_0) such that the IVP has more than one solution. [6]

Solution:

(a) Here $f(x,y) = 2(x-1)y/(x^2-2x)$ and $\partial f/\partial y = 2(x-1)/(x^2-2x)$. The existence and uniqueness theorem guarantees the existence of unique solution in the vicinity of (x_0, y_0) where f and $\partial f/\partial y$ are continuous and bounded. Thus, existence of unique solution is guaranteed at all x_0 for which $x_0(x_0-2) \neq 0$. Hence, unique solution exists when $x_0 \neq 0, 2$.

[2]

When $x_0 = 0$ or $x_0 = 2$, nothing can be said using the existence and uniqueness theorem. However, since the equation is separable, we can find the general solution to be $y = Cx(x-2)$.

[2]

Using initial condition we get $y_0 = Cx_0(x_0-2)$. Clearly the IVP has no solution if $x_0(x_0-2) = 0$ and $y_0 \neq 0$. [1]

If $x_0(x_0-2) = 0$ and $y_0 = 0$ then $y = \alpha x(x-2)$ is a solution to the IVP for any real α. [1]

Hence, in summary
(i) No solution for $x_0 = 0$ or $x_0 = 2$ and $y_0 \neq 0$;
(ii) Infinite number of solutions for $x_0 = 0$ or $x_0 = 2$ and $y_0 = 0$;
(iii) Unique solution for $x_0 \neq 0, 2$.