QB: A

Name: Roll No: Section:

Quiz-I MTH-102A
LINEAR ALGEBRA AND ODE
Spring-2018
Date: 28th January 2019

Time: 6.30 PM-7.00 PM Max. Marks: 20

1. Write your answer only in the space provided and explain all the major steps.

2. No additional sheets will be provided. Rough work may be done in the space provided at the end.

3. Unless explicitly stated otherwise, all vector spaces will be assumed to be over the set of real numbers $\mathbb{R}$.

Q1. a. The dimension of the vector space of all 2×2 real matrices of trace 0 is $\underline{\mathbf{3}}$. [1]

b. Let A be a 4×4 matrix with determinant 6. Then $det(E_{23}(-3)A) = \underline{\mathbf{6}}$. [1]

c. True or False: If a system $Ax = d$ has a unique solution, then A must be a square matrix. **False**. [1]

d. True or False: For a subspace W of a vector space V if $u, v \notin W$ then $u + v \notin W$. [1]
False

e. Let A be a 4×3 matrix with all entries equal to $\sqrt{2}$. Then the row reduced echelon form of A is [1]

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Q2. Let V be the space of all 2×2 matrices with entries in $\mathbb{R}$. Find a basis $\{A_1, A_2, A_3, A_4\}$ of V such that $A_i^2 = A_i$ for each i. Give proper justification. [5]

Solution 1: Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$.

Then $A_i^2 = A_i$ for each i. [1]

Linear Independence: If $a.A_1 + b.A_2 + c.A_3 + d.A_4 = 0$ then we have $c = d = 0$, $a + c = 0$ and $b + d = 0$. So $a = b = c = d = 0$. [2]

Again $\begin{bmatrix} x & y \\ z & t \end{bmatrix} = (x - y) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (t - z) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + y \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} + z \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$. [2]

So A_1, A_2, A_3 and A_4 spans V.

Note: Giving one or two correct basis elements doesn't carry any marks.

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Solution 2: Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, $A_4 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

Then $A_i^2 = 4A_i$ for each i. [1]

Linear Independence: If $a.A_1 + b.A_2 + c.A_3 + d.A_4 = 0$ then we have $c = d = 0$, $a + c = 0$ and $b + d = 0$. So $a = b = c = d = 0$. [2]

Again $\begin{bmatrix} x & y \\ z & t \end{bmatrix} = (x - z) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (t - y) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + z \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + y \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$. [2]

So A_1, A_2, A_3 and A_4 spans V.

Note: Giving one or two correct basis elements doesn't carry any marks.

Q3. Using **the definition** of determinant show that $det(A) = det(A^T)$. [5]

Solution: Let $B = A^T$. Then $b_{ij} = a_{ji}$ for all i and j. [0]

By definition $det(B) = \sum_{\sigma \in S_n} Sgn(\sigma) b_{1\sigma(1)} b_{2\sigma(2)} \cdots b_{n\sigma(n)}$.

$= \sum_{\sigma \in S_n} Sgn(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n}$ [1]

If $\sigma(i) = j$ then $i = \sigma^{-1}(j)$. So $a_{\sigma(1)1} a_{\sigma(2)2} \cdots a_{\sigma(n)n} = a_{1\sigma^{-1}(1)} a_{2\sigma^{-1}(2)} \cdots a_{n\sigma^{-1}(n)}$.

So $det(B) = \sum_{\sigma \in S_n} Sgn(\sigma) a_{1\sigma^{-1}(1)} a_{2\sigma^{-1}(2)} \cdots a_{n\sigma^{-1}(n)}$. [2]

$= \sum_{\sigma \in S_n} Sgn(\sigma^{-1}) a_{1\sigma^{-1}(1)} a_{2\sigma^{-1}(2)} \cdots a_{n\sigma^{-1}(n)}$, since $Sgn(\sigma^{-1}) = Sgn(\sigma)$ [1]

$= \sum_{\sigma \in S_n} Sgn(\sigma^{-1}) a_{1\sigma^{-1}(1)} a_{2\sigma^{-1}(2)} \cdots a_{n\sigma^{-1}(n)} = det(A)$, since $S_n = \{\sigma^{-1} : \sigma \in S_n\}$. [1]

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Note that for a $n \times n$ matrix A by **the definition** we have
$\det(A) = \sum_{\sigma \in S_n} Sgn(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$.

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Note that this is **THE SOLUTION** of this question using **the definition** and no other forms of solutions will be accepted.

Q4. Let V be the vector space of functions from $\mathbb{R}$ to $\mathbb{R}$. Find a basis for the subspace $W = Span\{1, sinx, sin^2x, cos^2x\}$. Give proper justification. [5]

Solutions 1: $B = \{1, sinx, sin^2x\}$. [1]

Linear Independence: Suppose $a.1 + b.sinx + c.sin^2x = 0$. [0]

If $x = 0$ then $a = 0$

If $x = \frac{\pi}{2}$ then $a + b + c = 0$

If $x = -\frac{\pi}{2}$ then $a - b + c = 0$ [2]

From these equations we get $a = b = c = 0$. So $B = \{1, sinx, sin^2x\}$ is LI. [1]

Since $cos^2x = 1 - sin^2x$, the set B spans W. [1]

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Solutions 2: $B = \{1, sinx, cos^2x\}$. [1]

Linear Independence: Suppose $a.1 + b.sinx + c.cos^2x = 0$. [0]

If $x = 0$ then $a + c = 0$

If $x = \frac{\pi}{2}$ then $a + b = 0$

If $x = -\frac{\pi}{2}$ then $a - b = 0$ [2]

From these equations we get $a = b = c = 0$. So $B = \{1, sinx, cos^2x\}$ is LI. [1]

Since $sin^2x = 1 - cos^2x$, the set B spans W. [1]

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Solutions 3: $B = \{sinx, sin^2x, cos^2x\}$. [1]

Linear Independence: Suppose $a.sinx + b.sin^2x + c.cos^2x = 0$. [0]

If $x = 0$ then $c = 0$

If $x = \frac{\pi}{2}$ then $a + b = 0$

If $x = -\frac{\pi}{2}$ then $-a + b = 0$ [2]

From these equations we get $a = b = c = 0$. So $B = \{sinx, sin^2x, cos^2x\}$ is LI. [1]

Since $1 = sin^2x + cos^2x$, the set B spans W. [1]