Research Statement

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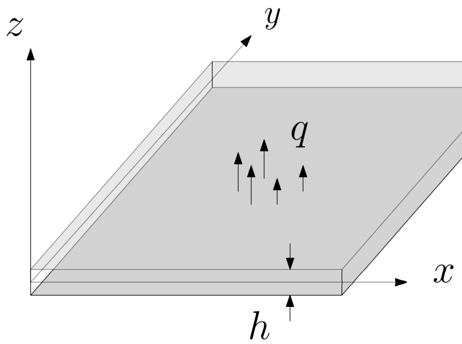
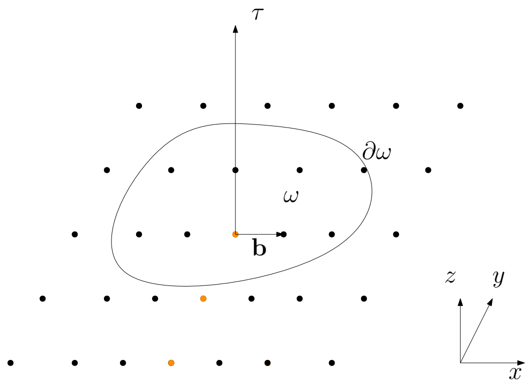
Defects in Anisotropic Plates

The aim of our work is to study the mechanical behaviour of slender structures such as plates, rods and shells in the presence of defects, such as disclinations, dislocations, interstitials and vacancies, and metric anomalies, such as thermal and growth strains. The premise is taken to be a thin elastic sheet, which is modeled as a Föppl–von Kármán plate, allowing for moderately large deflections out of the plane of the plate [1]. The schematic of a plate and a plate with a defect (an edge dislocation here) is given in Fig. 1.



(a) Plate Schematic.



(b) Plate with an Edge Dislocation.

Figure 1: Plate.

We use a mixed formulation of the Föppl–von Kármán theory, with the Airy stress function (χ) and the out-of plane deflection (w) as our variables, instead of the displacement formulation (u, v, w; u, v being the in-plane displacements and w as before). This saves us computation time with the finite element analysis, and also allows us to calculate the in-plane stresses directly from χ. The governing equations, for a particular case of anisotropy: the monolayered orthotropic plate, are given as [2],

$$\frac{h^3}{12}C_{11}w_{,xxxx} + \frac{h^3}{12}C_{22}w_{,yyyy} + \frac{h^3}{6}(C_{12}+C_{66})w_{,xxyy} - [w,\chi] = \Omega, \tag{1}$$

$$\frac{[w,w]}{2} + \frac{1}{h}[2(c_{66}+c_{12})\chi_{,xxyy} + c_{11}\chi_{,yyyy} + c_{22}\chi_{,xxxx}] = \lambda, \tag{2}$$

where C_{11}, C_{12}, C_{22} and C_{66} are the material parameters (the ones in small-case letters are components of the inverse of the stiffness matrix). λ and Ω are source terms which include information about the aforementioned defects and metric anomalies. Eq.(1) is obtained from equilibrium in the direction normal to the plate, and Eq.(2) is obtained from equilibrium in the plane of the plate (which allows us to write the stresses in terms of the Airy stress function), and the strain compatibility equation. The above equations reduce down to those pertaining to the isotropic case upon using correct expressions of the material parameters in terms of the Young's modulus E and the Poisson's ratio ν. "$[\cdot,\cdot]$" is

$$[\Phi,\Psi] = \{\Phi_{,xx}\Psi_{,yy} + \Psi_{,xx}\Phi_{,yy} - 2\Phi_{,xy}\Psi_{,xy}\}. \tag{3}$$

The stresses are given in terms of the Airy stress function as

$$h\sigma_x = \frac{\partial^2\chi}{\partial y^2}, \quad h\sigma_y = \frac{\partial^2\chi}{\partial x^2}, \quad h\sigma_{xy} = -\frac{\partial^2\chi}{\partial x\partial y}. \tag{4}$$

In the subsequent sections, we discuss the FEM formulation of our problem and a few of the early results obtained using the same for a positive disclination ($\lambda = s\delta_0$ and $\Omega = 0$, $s\delta_0$ being the Dirac-Delta Function with a peak at the center of the plate) in three types of plates: (i) isotropic, (ii) slightly anisotropic and (iii) heavily anisotropic.

FEM Formulation

For three cases of boundary conditions (free, simply supported and clamped [3]), we solve the boundary value problems using FEM, having developed our own MATLAB-based code from scratch. A mixed variational principle is used, according to which the governing equations appear as the stationary conditions of the functional (tweaked for anisotropy from [4], where it is given for the isotropic case)

$$\begin{aligned}\Pi(\chi, w) = &\int_\omega -\frac{1}{2h}[c_{11}\chi_{,yy}^2 + c_{22}\chi_{,xx}^2 + 2c_{12}\chi_{,yy}\chi_{,xx} + 2c_{66}\chi_{,xy}^2]dA \\ &+ \int_\omega \frac{h^3}{24}[C_{11}w_{,xx}^2 + C_{22}w_{,yy}^2 + 2C_{12}w_{,xx}w_{,yy} + 2C_{66}w_{,xy}^2]dA \\ &+ \int_\omega \frac{1}{2}[\chi_{,yy}w_{,x}^2 + \chi_{,xx}w_{,y}^2 - 2\chi_{,xy}w_{,x}w_{,y}]dA \\ &+ \int_\omega s\delta_0\chi dA,\end{aligned} \tag{5}$$

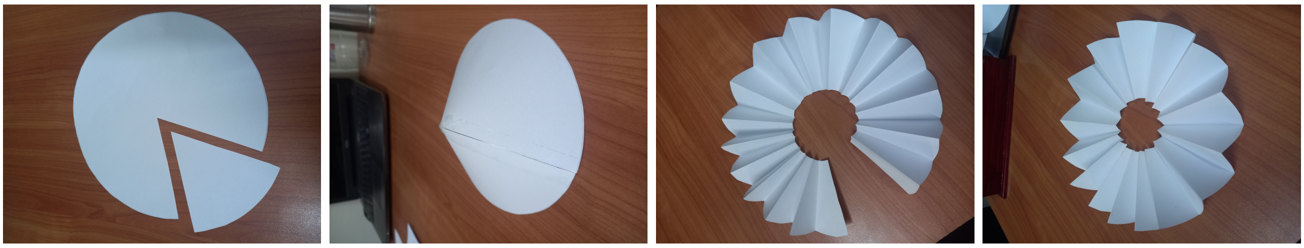
where ω is the domain of the plate (of length L, breadth B and height h). The rectangular plate domain is discretized using $N_x * N_y$ nonconforming C^1-continuous rectangular elements, and the weak form of the variational principle is used to obtain a system of nonlinear algebraic equations. The algebraic equations are solved using an arc-length method (Riks method). The equations are nonlinear and hence the solutions obtained are not unique. Different solution paths can be traced depending on the initial guess of the parameters involved in the numerical procedure. All the solutions are stationary points of the functional Π but not all are necessarily stable. The stable (metastable) solution corresponds to a point of global (local) minima in the strain energy landscape. The results in the next section are for a square plate with $L = B = 1$, divided into a mesh of 24×24 elements, with strength of the positive disclination, $s = \frac{\pi}{3}$. The three cases of material constants are as follows [5] (all units are MPa),

1. Isotropic: $C_{11} = C_{22} = 3325.43$, $C_{12} = 997.63$, $C_{66} = 2327.80$, $h = 0.033$ m.
2. Heavily Anisotropic: $C_{11} = 9830.05$, $C_{22} = 412.86$, $C_{12} = 98.30$, $C_{66} = 735.50$, $h = 0.01$ m.
3. Slightly Anisotropic: $C_{11} = 11798.14$, $C_{22} = 5899.07$, $C_{12} = 424.73$, $C_{66} = 686.47$, $h = 0.01$ m.

Results for a Positive Disclination

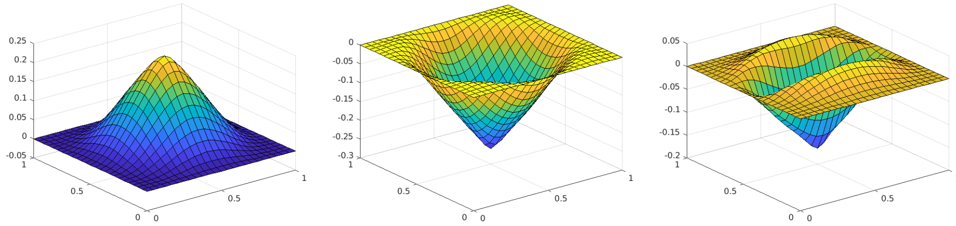
Although a variety of interesting shapes are possible with a positive disclination on a flat plate, we focus on the two important ones: the flat plate solution, and the perfect cone, which is formed due to buckling of the plate out of its plane. Paper models are shown for the same in Fig. 2. Fig. 3 shows how the conical shape differs according to the cases of material constants for the clamped boundary condition. The flat plate solution is not a stable one, and in all our cases of boundary conditions and material constants, is found to have a higher energy as compared to the conical (stable) solution, for which the bending energy (due to out of plane buckling) dominates the stretching energy (due to in-plane stretching). The flat plate solution has only stretching energy, as there is no bending involved.

Figs. 4, 5 and 6 show the distribution of the in-plane stresses σ_x, σ_y and σ_{xy} in the plate for the three cases of material constants and clamped boundary condition. The last figure (Fig. 7) shows the distribution of the Gaussian curvature (K), which is defined as $\frac{1}{2}[w, w]$, for the three cases. Gaussian curvature is a quantification of the curvature intrinsic to the surface, and not of how it is isometrically embedded in the 3D Euclidean space. It is also defined as the product of the two principal curvatures at a given point. A locally spherical surface will have a positive K, a flat plane or a cylindrical surface will have a zero K, and a saddle will have a negative K.



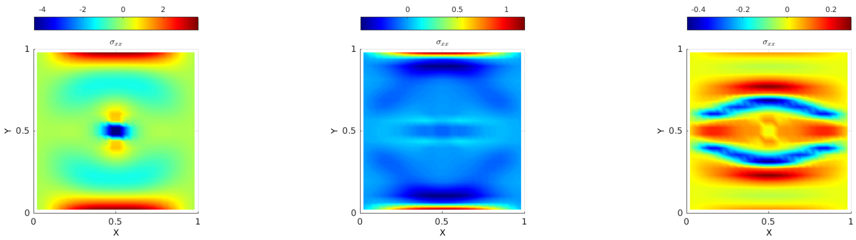
(a) (b) (c) (d)

Figure 2: (a) Wedge removed and the edges joined to get a (b) Conical shape. (c) Wedge removed from the (flat) corrugated (cylindrical anisotropy) annular sheet to get a (d) (stable) Flat shape.



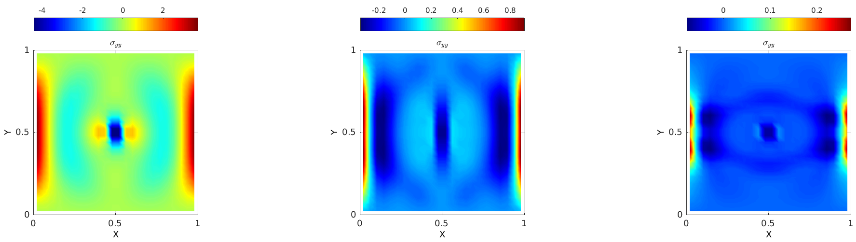
(a) (b) (c)

Figure 3: Conical solution with clamped boundary condition for (a) isotropic, (b) slightly anisotropic and (c) heavily anisotropic plate.



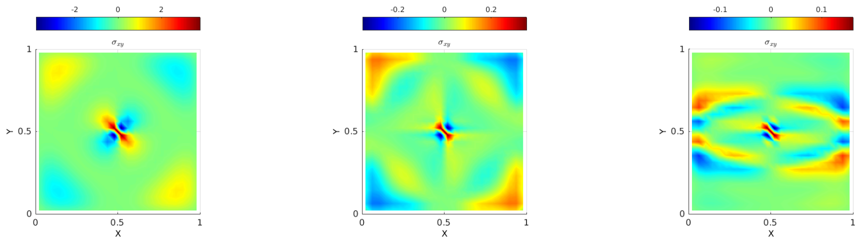
(a) (b) (c)

Figure 4: σ_x with clamped boundary condition for (a) isotropic, (b) slightly anisotropic and (c) heavily anisotropic plate.



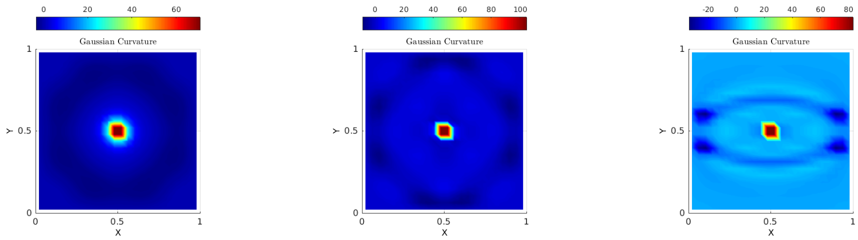
(a) (b) (c)

Figure 5: σ_y with clamped boundary condition for (a) isotropic, (b) slightly anisotropic and (c) heavily anisotropic plate.



(a) (b) (c)

Figure 6: σ_{xy} with clamped boundary condition for (a) isotropic, (b) slightly anisotropic and (c) heavily anisotropic plate.



(a) (b) (c)

Figure 7: Gaussian curvature with clamped boundary condition for (a) isotropic, (b) slightly anisotropic and (c) heavily anisotropic plate.

References

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