the bounding surface S of the volume V, then the field $\vec{A}$ is uniquely specified. Following this theorem, we can separate the given vector field into two parts:

$$\vec{A}(\vec{r}) = \vec{A}_D + \vec{A}_R \tag{2.27}$$

where $\vec{A}_D$ is an irrotational field with zero curl and non-zero divergence only, and $\vec{A}_R$ is a divergenceless field with a non-zero curl only such that $\nabla \cdot \vec{A} = \nabla \cdot \vec{A}_D$ and $\nabla \times \vec{A} = \nabla \times \vec{A}_R$. The Helmholtz theorem is of a great help in those situations where we know the divergence and the curl of a vector field. Then we can be sure that there is a unique vector field that has the divergence and the curl, subject to the specification of the boundary terms.

2.3 Curvilinear geometries and coordinates

In our discussions of electromagnetism in this course, very often we will deal with geometries containing cylinders or spheres. The Cartesian coordinate geometry is not the most well suited system to handle spherical and cylindrical geometries. Particularly, if there are symmetries associated with the problem such as an invariance with angle or distance from a given point, considerable simplifications can occur in the calculations if other coordinate systems are used. Usually it is simpler to consider coordinate systems with orthogonal axes. Here we will formally introduce and detail the three orthogonal coordinate systems that we will frequently use.

2.3.1 The Cartesian coordinate system

This is the familiar coordinate system to the student. Consider space in three dimensions: let us choose one point and call it the Origin. Now choose three mutually perpendicular axes in three dimensions that we will call as the X, Y, and the Z axes that intersect at the Origin (see Fig. 2.5). We label every point in space by three numbers, (x, y, z), that correspond to the distances from the origin that one would have to travel parallel to the three axes. Three unit vectors $(\hat{x}, \hat{y}, \hat{z})$ are defined in the directions along the three principal axes. Note that these unit vectors are constant vectors and remain the same when transposed to any given point – this follows from the property that the principal surfaces are planes in this coordinate system (see Fig. 2.5. By definition, we have $\hat{x} \cdot \hat{y} = \hat{y} \cdot \hat{z} = \hat{z} \cdot \hat{x} = 0$.

We list the following quantities for the sake of completeness and comparison with other coordinate systems:

1. The infinitesimal line element : $d\vec{r} = \hat{x}dx + \hat{y}dy + \hat{z}dz$.
2. The infinitesimal volume element: $d^3r = dx\ dy\ dz$.
3. The infinitesimal surface elements: $d\vec{s}_x = dy\ dz\ \hat{x}$, $d\vec{s}_y = dz\ dx\ \hat{y}$, $d\vec{s}_z = dx\ dy\ \hat{z}$.

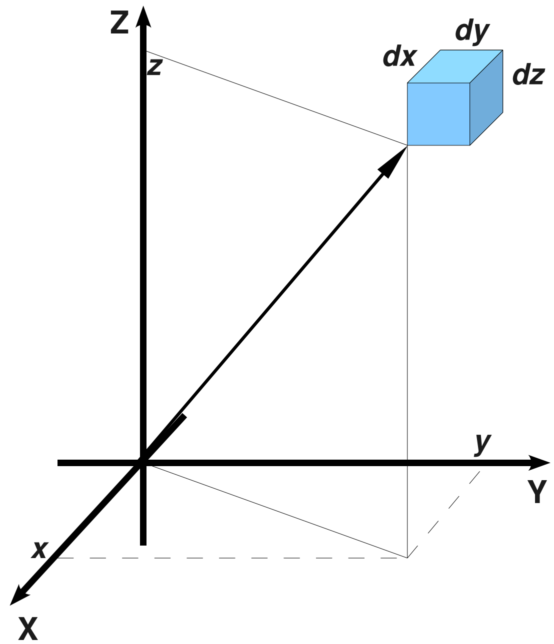

Figure 2.5 The Cartesian coordinate system consists of three mutually orthogonal axes along which the distances to a given point are measured. The infinitesimal volume element is also shown.

2.3.2 Cylindrical coordinate system

This becomes useful when the problem at hand has a preferred axis and when the fields primarily depend only on the absolute distance of the point from the the preferred axis. In this system, we label each point in space again by three numbers: but only two of them correspond to distances while the third corresponds to an angle. First we take the preferred axis (direction) and call it the Z axis. Choose the origin on this axis, and arbitrarily choose another direction in the plane perpendicular to the Z axis (– that would be the X–Y plane). Now any point in three dimensional space can be labelled by the radial distance (r) from the Z axis, the angle (ϕ) the radius vector makes with the X-axis in the X–Y plane and the

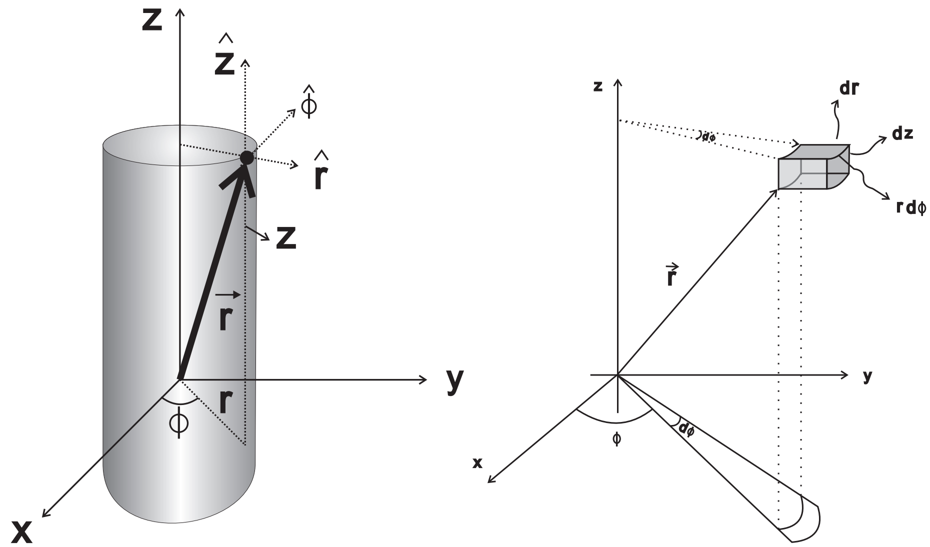

Figure 2.6 The Cylindrical coordinate system is a curvilinear coordinate system and is illustrated by the figure. The infinitesimal volume element is shown in the right panel.

height (Z) along the Z axis. This is depicted in Fig. 2.6. The values of these numbers are confined to the ranges $0 \leq r \leq \infty$, $0 \leq \phi < 2\pi$ and $-\infty < Z < \infty$ so that each point has a unique triplet that labels it. The relation to the Cartesian coordinates is obtained as

$$x = r\cos\phi, \qquad y = r\sin\phi, \qquad z = Z, \tag{2.28}$$

which are relations that can be easily inverted.

The unit vectors corresponding to each of these numbers point along the direction of increasing coordinate at each point as shown in Fig. 2.6. These are easily related to the Cartesian unit vectors as

$$\hat{r} = \cos\phi\ \hat{x} + \sin\phi\ \hat{y}, \tag{2.29}$$

$$\hat{\phi} = -\sin\phi\ \hat{x} + \cos\phi\ \hat{y}, \tag{2.30}$$

$$\hat{Z} = \hat{z}. \tag{2.31}$$

It can be easily verified that the unit vectors are mutually perpendicular $\hat{r} \cdot \hat{\phi} = \hat{\phi} \cdot \hat{Z} = \hat{Z} \cdot \hat{r} = 0$. It is clear from the above that the unit vectors change from point to point, in this case they depend on the location through the angle ϕ. This is unlike the unit vectors in the Cartesian system. Hence, we cannot thoughtlessly just move the unit vectors in or out across derivatives and integrals.

Another crucial difference comes from the consideration of the infinitesimal displacements along the three directions. Along the radial and axial directions, the infinitesimal displacements corresponds to the change in the coordinates (dr and dZ), which dimensions of length. Along the $\hat{\phi}$ direction, however, an infinitesimal change in the coordinate ($d\phi$) is an angle and translates to a length as $rd\phi$ (see Fig. 2.6). Thus, there is a scale factor of r that depends on the given point in space. The infinitesimal quantities in this coordinate system are:

1. The infinitesimal line element : $d\vec{r} = \hat{r}dr + \hat{\phi}rd\phi + \hat{Z}dZ$.
2. The infinitesimal volume element: $d^3r = dr\ rd\phi\ dZ$.
3. The infinitesimal surface elements: $d\vec{s}_r = rd\phi\ dZ\ \hat{r}$, $d\vec{s}_\phi = dZ\ dr\ \hat{\phi}$, $d\vec{s}_Z = dr\ rd\phi\ \hat{Z}$.

2.3.3 Spherical coordinates

When a given problem has complete angular symmetry, i.e., when no direction is preferrable over any other, the spherical coordinate system is very useful. Typically, all properties of the system depend only on the absolute distance from a specific point in space, which we will choose to be the origin. Now we will arbitrary choose an axis, the z axis and a x axis on the plane perpendicular to the z axis and containing the origin. Now any point in (three dimensional) space can be labelled uniquely by a triplet of numbers: one representing the absolute radial distance (r) to the origin, an angle θ indicating the angle between the radial line joining the origin to the given point and the chosen z axis, and another angle ϕ that is the angle between the projection of the radial line to the point on the X-Y plane and the chosen x axis (see Fig. 2.7). The values of these numbers are confined to the ranges $0 \le r \le \infty$, $0 \le \theta \le \pi$, and $0 \le \phi < 2\pi$ so that each point corresponds to a unique triplet that labels it. The relation to the Cartesian coordinates is obtained as

$$x = r\sin\theta\ \cos\phi, \quad y = r\sin\theta\ \sin\phi, \quad z = r\cos\theta, \tag{2.32}$$

relations that can be easily inverted as

$$r = (x^2+y^2+z^2)^{1/2}, \quad \theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right), \quad \phi = \tan^{-1}\left(\frac{y}{x}\right). \tag{2.33}$$

The unit vectors corresponding to each of these numbers point along the direction of increasing coordinate at each point as shown in Fig. 2.6. These are easily related to the Cartesian unit vectors as

$$\hat{r} = \sin\theta\cos\phi\ \hat{x} + \sin\theta\sin\phi\ \hat{y} + \cos\theta\ \hat{z}, \tag{2.34}$$

$$\hat{\theta} = \cos\theta\cos\phi\ \hat{x} + \cos\theta\sin\phi\ \hat{y} - \sin\theta\ \hat{z}, \tag{2.35}$$

$$\hat{\phi} = -\sin\phi\ \hat{x} + \cos\phi\ \hat{y}. \tag{2.36}$$

Once again note that these relations are invertible. As in the case of the cylindrical

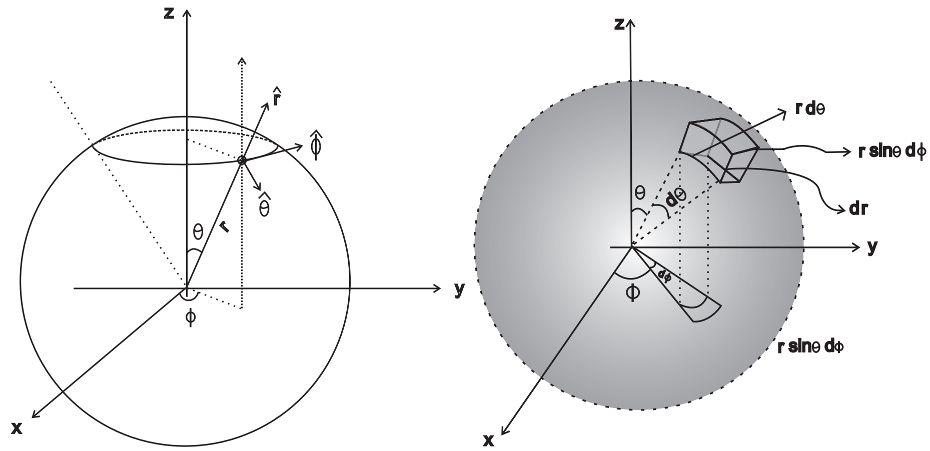

Figure 2.7 The Cartesian coordinate system consists of three mutually orthogonal axes along which the distances to a given point are measured. The infinitesimal volume element is also shown.

system, these unit vectors point in different directions at different points. This is a general property of all curvilinear coordinate systems which we will briefly discuss later. Hence one has to be careful while differentiating or integrating expressions containing these unit vectors. As examples consider the follwing two integrals:

$$\int_a^b \hat{r} dr = (b-a)\hat{r}, \ \int_0^{2\pi} \hat{r}\ d\phi = 0$$

In the first case, the $\hat{r}$ vector remains constant as r is changed, while it is not so in the second case.

The infinitesimal quantities in this system are:

1. The infinitesimal line element : $d\vec{r} = \hat{r}\ dr + \hat{\theta}\ rd\theta + \hat{\phi}\ r\sin\theta d\phi$.
2. The infinitesimal volume element: $d^3r = dr\ rd\theta\ r\sin\theta d\phi$.
3. The infinitesimal surface elements: $d\vec{s}_r = rd\theta\ r\sin\theta d\phi\ \hat{r}$, $d\vec{s}_\theta = dr\ r\sin\theta d\phi\ \hat{\theta}$, $d\vec{s}_\phi = dr\ rd\theta\ \hat{\phi}$.

It can be seen that the scale factor r multiplies the infinitesimal change $d\theta$ to give rise to an infinitesimal length $rd\theta$ along $\hat{\theta}$. Similarly, a scale factor $r\sin\theta$ (projected length of the radial vector on the X-Y plane) accompanies the infinitesimal quantity $d\phi$ to give an infinitesimal length $r\sin\theta d\phi$ along $\hat{\phi}$.

2.3.4 An orthogonal curvilinear coordinate system

We will not discuss in detail the properties of a curvilinear coordinate system, but will only list some results that can be written down in general for orthogonal coordinate systems. For details, we refer the reader to **?**. Consider an invertible mapping to the coordinate system (u_1, u_2, u_3) from the Cartesian coordinates by the functions:

$$u_1 = u_1(x, y, z), \quad u_2 = u_2(x, y, z), \quad u_3 = u_3(x, y, z). \tag{2.37}$$

It can be shown that the unit vectors are given by

$$\hat{u}_i = \frac{\nabla u_i}{|\nabla u_i|}. \tag{2.38}$$

The infinitesimal displacement vector can be written as

$$d\vec{r} = h_1 du_1 \ \hat{u}_1 + h_2 du_2 \ \hat{u}_2 + h_3 du_3 \ \hat{u}_3, \tag{2.39}$$

where the scale factors h_i are given by

$$h_i^2 = |\nabla u_i|^2. \tag{2.40}$$

Now the infinitesimal volume is written as

$$d^3r = h_1 du_1 \ h_2 du_2 \ h_3 du_3. \tag{2.41}$$

In general, we can also write down expressions for the gradient, divergence and curl in the generalized coordinates using the scale factors

$$\nabla f = \hat{u}_1 \frac{1}{h_1} \frac{\partial f}{\partial u_1} + \hat{u}_2 \frac{1}{h_2} \frac{\partial f}{\partial u_2} + \hat{u}_3 \frac{1}{h_3} \frac{\partial f}{\partial u_3}, \tag{2.42}$$

$$\nabla \cdot \vec{A} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial (A_1 h_2 h_3)}{\partial u_1} + \frac{\partial (A_2 h_3 h_1)}{\partial u_2} + \frac{\partial (A_3 h_1 h_2)}{\partial u_3} \right] \tag{2.43}$$

$$\nabla \times \vec{A} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{u}_1 & h_2 \hat{u}_2 & h_3 \hat{u}_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \tag{2.44}$$

It is easily seen that the scale factors for the cylindrical coordinates are given by

$$h_r = 1, \quad h_\phi = r, \quad h_Z = 1, \tag{2.45}$$

and for the spherical coordinates they are given by

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r \sin\theta. \tag{2.46}$$

Knowledge of the scale factors enables us to carry out all the calculations on the vectors fields in any desired coordinate system.

2.4 The Dirac δ− function

Consider the function

$$f(x) = \begin{cases} \frac{1}{2w} & \forall \ |x| < w \\ 0 & \forall \ |x| > w. \end{cases} \tag{2.47}$$

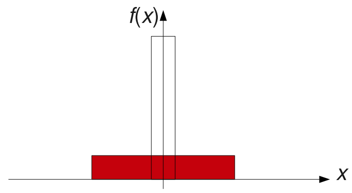

Figure 2.8 The rectangular function in the limit of (infinite) increasing height and simultaneously (zero) decreasing width while keeping the area constant behaves as a *delta*-function.

It is evident that the integral $\int_{-L}^{L} f(x)dx = 1$ if $L > w$ (see Fig. 2.8). Now examine this function in the limit $w \to 0$. It is clear that the function is zero everywhere except the single point $x = 0$ where it diverges, and yet the integral is exactly unity. This is contrary to our usual understanding of (Riemann) integrals where the value of the integral is zero unless the integration range is finite. In other words, a single point usually has zero *measure*. Yet this mathematical object that results from a well defined function in the limit $w \to 0$ has a non-zero integral.

We will often face such mathematical objects in our study of electromagnetism. Consider the following definition

$$\delta(x - x_0) = \begin{cases} 0 & \forall \;\; x \neq x_0, \\ \infty & \forall \;\; x = x_0, \end{cases} \tag{2.48}$$

such that the integral

$$\int_a^b \delta(x - x_0)dx = 1, \tag{2.49}$$

if the interval $[a, b]$ includes the singular point x_0 and is zero otherwise. It is simple to construct that $\delta(x - a) = \delta(a - x)$. Note that the principal properties of this object derive from the integral. The above mathematical construct was first formally discussed in connection with quantum mechanics by a scientist called Dirac, and it is called the Dirac δ function. Although we call this object a function, it is not a function in the conventional sense and belongs to a generalized class of functions called *distributions* by mathematicians.

The Dirac δ function can work as a sieve to pick out values of functions at specific points. It is easily seen that

$$\int_a^b \delta(x - x_0)f(x)dx = \lim_{\epsilon \to 0} \int_{x_0-\epsilon}^{x_0+\epsilon} \delta(x - x_0)f(x)dx = f(x_0), \tag{2.50}$$

where $f(x)$ is a usual continuous function. Sometimes it is convenient to work with certain functions that become δ functions in limiting cases. The rectangular function presented above is one example. Other possible examples include a Gaussian function

$$\delta(x - x_0) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right], \tag{2.51}$$

and a Lorentzian function

$$\delta(x - x_0) = \lim_{\sigma \to 0} \frac{1}{\pi} \frac{\sigma}{(x-x_0)^2 + \sigma^2}. \tag{2.52}$$

In both of the cases above, σ is linearly proportional to the full width of the functions where the value of the singly peaked functions falls to half their peak value. As this width falls to zero in the limit, the peak value rises keeping the value of the integral constant (unity).

The δ function can also be interpreted as the derivative of a step-function at the point of discontinuity. Consider the Heaviside step function defined as

$$\Theta(x - x_0) = \begin{cases} 1 & \forall \ x > x_0, \\ 0 & \forall \ x < x_0. \end{cases} \tag{2.53}$$

The vaue of the step function depends on whether its argument is positive or negative. Its derivative can be shown to be a δ function. We can do this by showing that it has the property of the δ function. The derivative is clearly zero everywhere except at x_0, and for an arbitrary function that is continuous at x_0, the integral

$$\begin{aligned} \int_a^b f(x) \frac{d}{dx} \Theta(x - x_0) \ dx &= [f(x)\Theta(x - x_0)]_a^b - \int_a^b \Theta(x - x_0) \frac{df}{dx} dx, \\ &= f(b) - \int_{x_0}^b \frac{df}{dx} dx, \\ &= f(b) - [f(b) - f(x_0)] = f(x_0), \end{aligned} \tag{2.54}$$

where the interval $[a, b]$ is assumed to contain the point x_0, and we have integrated by parts. Clearly the derivative of the step-function has all the essential properties of the δ function.

The idea of the δ function as a point of singularity but with a finite integral is easily extended to higher dimensions. In three dimensional space, we have the integral

$$\int_V \delta(\vec{r} - \vec{r_0}) d^3r = 1, \tag{2.55}$$

if the integration volume V contains the point $\vec{r_0}$ and is zero otherwise. In Cartesian coordinates, it is straightforward to represent the δ function as a product of one-dimensional δ functions,

$$\delta(\vec{r} - \vec{r_0}) = \delta(x - x_0) \ \delta(y - y_0) \ \delta(z - z_0). \tag{2.56}$$

Representation of higher dimensional δ functions in other co-ordinate systems will be discussed in the next section. Note that the one dimensional δ function has dimensions of inverse length, Hence the three-dimensional δ function has dimensions of inverse volume. This gives rise to the interpretation that the δ function is effectively a density. It is customary for some authors **?** to indicate the dimensionality of the space in which the δ function is defined, by a superscript: for example, $\delta^{(3)}(\vec{r} - \vec{r}_0)$. We will, however, only write it as $\delta(\vec{r} - \vec{r}_0)$, with the understanding that the dimensionality of the space is given by that in which the argument vectors are defined.

In general curvilinear coordinate systems, the infinitesimal volumes depend on the location of the point and it becomes important to normalize the δ function to account for this change in the density. For a δ function located on the point (u_1', u_2', u_3'), we write

$$\delta(\vec{r} - \vec{r'}) = \frac{1}{h_1 h_2 h_3}\delta(u_1 - u_1')\delta(u_2 - u_2')\delta(u_3 - u_3'). \tag{2.57}$$

Unless properly normalized, the δ function would begin to have different weights depending on where it is placed. Overall the δ function should be defined such that the integral over a volume containing the point where the singularity is located should yield unity. Thus, in spherical coordinates the δ function would be written as

$$\delta(\vec{r} - \vec{r'}) = \frac{1}{r^2 \sin\theta}\delta(r - r')\delta(\theta - \theta')\delta(\phi - \phi'). \tag{2.58}$$

Special mention must be made of points of singularity such as the origin or points on the z axis where the spherical coordinates θ and ϕ may become ill-defined, i.e., the point is multiply described by the curvilinear coordinates. In such cases, if the coordinate u_3 multiply describes the point where the δ function is located, there will be no such factor such as $\delta(u_3 - u_3')$ in the representation for the δ function, since the value of u_3' would be non-unique and ill-defined. Hence the representation the curvilinear coordinate system would only appear as

$$\delta(\vec{r} - \vec{r'}) = \frac{1}{h_1 h_2 \int_a^b h_3 du_3}\delta(u_1 - u_1')\delta(u_2 - u_2'). \tag{2.59}$$

Similar arguments would apply if two of the coordinates were multiply valued at a given point, for example, the origin in the spherical coordinate system. Then we have

$$\delta(\vec{r} - \vec{r'}) = \frac{1}{\int_p^q h_2 du_2 \ \int_a^b h_3 du_3}\delta(u - u_1). \tag{2.60}$$

EXAMPLES:

1. Consider a point charge q located at the origin. In cylindrical coordinates, the corresponding charge density would be described as

$$\delta(\vec{r}) = q\frac{1}{2\pi r}\delta(r)\delta(z).$$

In spherical coordinates, the representation would be

$$\delta(\vec{r}) = q\frac{1}{4\pi r^2}\delta(r).$$

2. Consider a charged thin disk of radius R carrying a charge per unit area of σ lying on the $X - Y$ plane. The volume charge density can be represented in cylindrical coordinates as

$$\rho(\vec{r}) = \sigma\delta(z)\Theta(R - r),$$

while the representation in spherical coordinates is

$$\rho(\vec{r}) = \sigma\frac{1}{r}\delta(\theta - \pi/2)\Theta(R - r).$$

Note that the Heaviside step function has been used to confine the charge toa radius smaller than R.

3. Consider a line charge with linear charge density λ per unit length, located along the Z axis. In Cartesian coordinates, this is is easily represented as

$$\rho(\vec{r}) = \lambda\delta(x)\delta(y),$$

while in the cylindrical coordinates, we can write

$$\rho(\vec{r}) = \lambda\frac{1}{2\pi r}\delta(r),$$

and in the spherical coordinates, the representation would be

$$\rho(\vec{r}) = \lambda\frac{1}{2\pi r^2 \sin\theta}[\delta(\theta) + \delta(\theta - \pi)].$$