Notes for PHY103N: Physics II

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The uniqueness theorem for electrostatic fields

The potential V in the region of interest is governed by the Poisson equation,

$$\nabla^2 V(\vec{r}) = -\frac{\rho(\vec{r})}{\varepsilon_0}, \tag{1}$$

where $\rho(\vec{r})$ is the volume charge density which could be spatially varying, and the conditions on the potential have to be specified on the boundaries S of the region of interest. These could be in the form of the potential itself $V(\vec{r}) = f(\vec{r})$ on S or could be the normal derivative $(\partial V/\partial n) = g(\vec{r})$ on S (sepcifying the normal derivative is equivalent to specifying the surface charge density). Here f and g are some specified functions.

Now if there could be two different solutions within the volume enclosed by S, which satisfy the Poisson equation and the above boundary conditions, let them be $V_1(\vec{r})$ and $V_2(\vec{r})$. We can define

$$\chi = V_1(\vec{r}) - V_2(\vec{r}).$$

Now, we have

$$\nabla^2\chi = 0 \text{ and } \chi = 0 \text{ on } S \quad \text{OR} \quad \frac{\partial\chi}{\partial n} = 0 \text{ on } S \tag{2}$$

Let us consider the vector $\phi\ \nabla\psi$ where ϕ and ψ are some arbitrary scalar functions of $\vec{r}$. Using the Divergence theorem, we have

$$\int_{\mathcal{V}} \nabla\cdot(\phi\nabla\psi)\mathrm{d}r^3 = \oint_S \phi\frac{\partial\psi}{\partial n}\mathrm{d}s. \tag{3}$$

Now using the product rule for the divergence,

$$\int_{\mathcal{V}} \left(\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi\right)\mathrm{d}r^3 = \oint_S \phi\frac{\partial\psi}{\partial n}\mathrm{d}s \tag{4}$$

Let us now substitute $\phi = \psi = \chi$ and obtain,

$$\int_{\mathcal{V}} \left[\chi\nabla^2\chi + (\nabla\chi)^2\right]\mathrm{d}r^3 = \oint_S \chi\frac{\partial\chi}{\partial n}\mathrm{d}s.$$

Substituting in for $\nabla^2\chi$ and the values of χor its normal derivative on S, we obtain

$$\int_{\mathcal{V}} (\nabla\chi)^2 \mathrm{d}r^3 = 0 \tag{5}$$

This implies that $\nabla\chi = 0 \quad \Rightarrow \quad \chi = \text{constant} = 0$ if V is specified on the boundary, or $V_1 = V_2$. In the second case, when the normal derivative is specified on S, we have that V_1 and V_2 differ at most by a finite constant. Thus, the solution to the Poisson equation under these conditions is unique.

When the potential V is specified on S, it is termed as *Dirichlet boundary conditions* and when the normal derivative $(\partial V/\partial n)$ is specified on the boundaries S, we call these as the *Neumann boundary conditions*. It is possible to have *Mixed boundary conditions* where V is specified over some parts of S and $(\partial V/\partial n)$ on the others. But it is not possible to specify both V and $(\partial V/\partial n)$ on the same part of the boundaries.