MSO202A: Assignment-II

Notation: $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$

1. If f is differentiable in an open set Ω, then

$$\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) = 0 \quad \text{and} \quad f'(z) = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

2. Let $f(z) = z^3$. For $z_1 = 1$ and $z_2 = i$, show that there does not exist any point c on the line $x + y = 1$ joining z_1 and z_2 such that $f(z_1) - f(z_2) = (z_1 - z_2)f'(c)$, i.e., mean value theorem does not extend to complex plane.

3. Derive C-R equations in polar coordinates.

4. Let $f : \mathbb{D} \to \mathbb{C}$ be a differentiable function such that, for all $z, w \in \mathbb{C}$, $f(z) = f(w)$ whenever $|z| = |w|$. Using CR equations in polar coordinates, show that f is a constant function.

5. Show that the function $f(z) = \bar{z}$ is not differentiable at any point of $\mathbb{C}$. Find the points of differentiability of the function $f_a(z) = (z - a)\text{Re}(z - a)$ for a given $a \in \mathbb{C}$.

6. Let U be an open set and $f : U \to \mathbb{C}$ be a differentiable function. Let $\bar{U} = \{\bar{z} : z \in \mathbb{C}\}$. Show that $g : \bar{U} \to \mathbb{C}$ defined by $g(z) := \overline{f(\bar{z})}$ is differentiable on $\bar{U}$.

7. Show that the following functions satisfy CR equations at $z = 0$, but they are not differentiable at $z = 0$.
   (a)
$$f(z) = \begin{cases} \frac{z^5}{|z|4} & \text{if} \quad z \neq 0 \\ 0 & \text{if} \quad z = 0. \end{cases}$$
   (b) $f(z) = \sqrt{|xy|}$

8. Let Ω be an open connected subset of $\mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be a differentiable function. Show that $f = u + iv$ is constant if
   (a) either of the functions u or v is constant, or
   (b) $|f(z)|$ is constant for all $z \in \Omega$, or
   (c) if there exists an $\alpha \in \mathbb{R}$ such that $f(z) = |f(z)|e^{i\alpha}$ for all $z \in \Omega$

9. Show that the function $f(z) = (2 - x^2 - y^2)(x - iy)$ has derivative only on the points of the circle $x^2 + y^2 = 1$.

10. Does there exist an analytic function $f(z) = u + iv$ where $u(x, y)$ is given by (a) x^2y (b) $e^x \cos(x - y)$ (c) $e^x \sin y$

11. If $f(z)$ is an analytic function, then show that $\nabla^2|f(z)|^2 = 4|f'(z)|^2$.

12. Find the domain in which the function $f(z) = |\text{Re } z^2| + i|\text{Im } z^2|$ is analytic.